5. [16 points] Identify each of the following as true or false. Give a one-sentence explanation for your response in each case.

a. [4 points] Euler's method applied to the system $\mathbf{x}' = \begin{pmatrix} t & 0 \\ 1 & t^2 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives, after 2 steps with $h = 0.5$, $\mathbf{x}(1) \approx \begin{pmatrix} 0 \\ 1.25 \end{pmatrix}$.

True **[False]**

Solution: Because $\mathbf{x}'(0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{0}$, Euler's method gives $\mathbf{x}(0.5) \approx \mathbf{x}(0)$; then $\mathbf{x}'(0.5) \approx \begin{pmatrix} 0.5 & 0 \\ 1 & 0.25 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.25 \end{pmatrix}$, and $\mathbf{x}(1) \approx \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 0.5 \begin{pmatrix} 0 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 0 \\ 1.125 \end{pmatrix}$.

b. [4 points] Given that $\mathbf{x}_1 = \begin{pmatrix} e^t \\ 3e^t \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 2e^t \\ 6e^t \end{pmatrix}$ are solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for some 2×2 matrix $\mathbf{A}$, a general solution is $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$.

True **[False]**

Solution: The two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are not linearly independent, so this cannot be a general solution.

c. [4 points] If $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\ldots$, $\mathbf{x}_n(t)$ are solutions to a system of n linear first-order differential equations, and if $\mathbf{x}_1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\mathbf{x}_2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\ldots$, $\mathbf{x}_n(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$, then a general solution to the system is given by $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$.

[True] False

Solution: Note that $W[\mathbf{x}_1(0), \ldots, \mathbf{x}_n(0)] = 1$; thus the $\mathbf{x}_j$ are linearly independent and the general solution is as given.

d. [4 points] If one or more of the eigenvalues of the constant matrix $\mathbf{A}$ are zero, the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ has no solution.

True **[False]**

Solution: Because of the existence theorem we know that there is a solution.