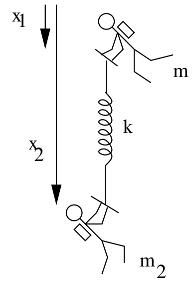
7. [15 points] The figure to the right shows two (hypothetical) skydivers, with a spring connecting them. We assume that the mass of the first, m_1, is less than the mass of the second, m_2. The distances that each has fallen are x_1 and x_2, and the spring constant is k. Let L be the equilibrium length of the spring. Then the system is modeled as



$$x_1'' = \frac{k}{m_1}(-x_1 + x_2) + \left(g - \frac{kL}{m_1}\right)$$
$$x_2'' = \frac{k}{m_2}(x_1 - x_2) + \left(g + \frac{kL}{m_2}\right).$$

a. [3 points] If we write this as a matrix equation $\mathbf{x}'' = \mathbf{A}\mathbf{x} + \mathbf{f}$, what are $\mathbf{x}$, $\mathbf{A}$ and $\mathbf{f}$?

Solution: These are

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -k/m_1 & k/m_1 \\ k/m_2 & -k/m_2 \end{pmatrix}, \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} g - kL/m_1 \\ g + kL/m_2 \end{pmatrix}.$$

b. [4 points] Now suppose that we're interested in finding the solution to the homogeneous problem associated with this system. If we take $\mathbf{x} = \mathbf{v}e^{\omega t}$, what equation must $\mathbf{v}$ and ω satisfy? How are $\mathbf{v}$ and ω related to the matrix $\mathbf{A}$ that you found above?

Solution: Plugging $\mathbf{x} = \mathbf{v}e^{\omega t}$ into the homogeneous problem $\mathbf{x}'' = \mathbf{A}\mathbf{x}$, we get $\omega^2\mathbf{v}e^{\omega t} = \mathbf{A}\mathbf{v}e^{\omega t}$. We may divide out the exponential to obtain $\omega^2\mathbf{v} = \mathbf{A}\mathbf{v}$, so that $\omega^2 = \lambda$, the eigenvalues of $\mathbf{A}$, and $\mathbf{v}$ are the corresponding eigenvectors.

c. [8 points] Now suppose that the eigenvalues and eigenvectors of the matrix $\mathbf{A}$ you found in (a) are $\lambda_1 = 0$, with $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = -4$ with $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Write the complementary homogeneous solution to your system.

Solution: From the work above, if $\lambda = 0$ we know that $\omega = 0$ (twice), and we get the two solutions $\mathbf{x}_1 = \mathbf{v}_1$ and $\mathbf{x}_2 = \mathbf{v}_1 t$. If $\lambda = -4$ we have $\omega = \pm 2i$, so that we gain the two additional solutions $\mathbf{x}_3 = \mathbf{v}_2\cos(2t)$ and $\mathbf{x}_4 = \mathbf{v}_2\sin(2t)$. The general solution is therefore

$$\mathbf{x} = (c_1 + c_2 t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (c_3\cos(2t) + c_4\sin(2t))\begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$