7. [15 points] The figure to the right shows two (hypothetical) skydivers, with a spring connecting them. We assume that the mass of the first, \( m_1 \), is less than the mass of the second, \( m_2 \). The distances that each has fallen are \( x_1 \) and \( x_2 \), and the spring constant is \( k \). Let \( L \) be the equilibrium length of the spring. Then the system is modeled as

\[
x_1'' = \frac{k}{m_1} (-x_1 + x_2) + \left( g - \frac{kL}{m_1} \right)
\]

\[
x_2'' = \frac{k}{m_2} (x_1 - x_2) + \left( g + \frac{kL}{m_2} \right).
\]

a. [3 points] If we write this as a matrix equation \( \mathbf{x}'' = \mathbf{A} \mathbf{x} + \mathbf{f} \), what are \( \mathbf{x} \), \( \mathbf{A} \) and \( \mathbf{f} \)?

Solution: These are

\[
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -\frac{k}{m_1} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} \end{pmatrix}, \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} g - \frac{kL}{m_1} \\ g + \frac{kL}{m_2} \end{pmatrix}.
\]

b. [4 points] Now suppose that we’re interested in finding the solution to the homogeneous problem associated with this system. If we take \( \mathbf{x} = \mathbf{v} e^{\omega t} \), what equation must \( \mathbf{v} \) and \( \omega \) satisfy? How are \( \mathbf{v} \) and \( \omega \) related to the matrix \( \mathbf{A} \) that you found above?

Solution: Plugging \( \mathbf{x} = \mathbf{v} e^{\omega t} \) into the homogeneous problem \( \mathbf{x}'' = \mathbf{A} \mathbf{x} \), we get \( \omega^2 \mathbf{v} e^{\omega t} = \mathbf{A} \mathbf{v} e^{\omega t} \). We may divide out the exponential to obtain \( \omega^2 \mathbf{v} = \mathbf{A} \mathbf{v} \), so that \( \omega^2 = \lambda \), the eigenvalues of \( \mathbf{A} \), and \( \mathbf{v} \) are the corresponding eigenvectors.

c. [8 points] Now suppose that the eigenvalues and eigenvectors of the matrix \( \mathbf{A} \) you found in (a) are \( \lambda_1 = 0 \), with \( \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( \lambda_2 = -4 \) with \( \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \). Write the complementary homogeneous solution to your system.

Solution: From the work above, if \( \lambda = 0 \) we know that \( \omega = 0 \) (twice), and we get the two solutions \( \mathbf{x}_1 = \mathbf{v}_1 \) and \( \mathbf{x}_2 = \mathbf{v}_1 t \). If \( \lambda = -4 \) we have \( \omega = \pm 2i \), so that we gain the two additional solutions \( \mathbf{x}_3 = \mathbf{v}_2 \cos(2t) \) and \( \mathbf{x}_4 = \mathbf{v}_2 \sin(2t) \). The general solution is therefore

\[
\mathbf{x} = (c_1 + c_2 t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (c_3 \cos(2t) + c_4 \sin(2t)) \begin{pmatrix} 3 \\ 1 \end{pmatrix}.
\]