2. [14 points] Find each of the following. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] $\mathcal{L}\{f(t)\}$, if $f(t) = \begin{cases} 1-t, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$

Solution: Note that $f(t) = (1-t)(1-u_1(t)) = -(t-1) + (t-1)u_1(t)$. From our table of transforms, we note that $\mathcal{L}\{t\} = \frac{1}{s^2}$ and $\mathcal{L}\{(t-1)\} = \frac{1}{s^2} - \frac{1}{s}$. Thus

$$\mathcal{L}\{f(t)\} = -\left(\frac{1}{s^2} - \frac{1}{s}\right) + \frac{1}{s^2}\,e^{-s} = \frac{1}{s} + \frac{1}{s^2}\left(e^{-s} - 1\right).$$

Alternately, we can find this by direct integration:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\,e^{-st}\,dt = \int_0^1 (1-t)e^{-st}\,dt = -\frac{1}{s}\,e^{-st}\Big|_{t=0}^{t=1} - \int_0^1 t\,e^{-st}\,dt.$$

The remaining integral we complete by parts with $u = t$ and $v' = e^{-st}$, so that $\int t\,e^{-st}\,dt = -\frac{1}{s}\,t\,e^{-st} + \int \frac{1}{s}\,e^{-st}\,dt = -\frac{1}{s}\,t\,e^{-st} - \frac{1}{s^2}\,e^{-st}$. Thus

$$\mathcal{L}\{f(t)\} = \left(-\frac{1}{s}\,e^{-st} + \frac{1}{s}\,t\,e^{-st} + \frac{1}{s^2}\,e^{-st}\right)\Big|_{t=0}^{t=1} = \frac{1}{s} + \frac{1}{s^2}\left(e^{-s} - 1\right).$$

b. [7 points] $Y(s) = \mathcal{L}\{y(t)\}$, if $y'' + 9y = u_\pi(t)\cos(4(t-\pi))$, $y(0) = 1$, $y'(0) = 2$.

Solution: Transforming, we have $\mathcal{L}\{y'' + 9y\} = \mathcal{L}\{u_\pi(t)\cos(4(t-\pi))\}$, so that

$$s^2\,Y - s - 2 + 9Y = \frac{s\,e^{-\pi s}}{s^2 + 16},$$

and

$$Y = \frac{s+2}{s^2+9} + \frac{s\,e^{-\pi s}}{(s^2+9)(s^2+16)}.$$