3. [15 points] A chemical reaction with two reagents (chemicals) in amounts r_1 and r_2 that may be converted from one to the other may be modeled the system of first-order differential equations

$$r_1' = -3r_1 + 9r_2$$
$$r_2' = k\,r_1 - r_2 + f(t),$$

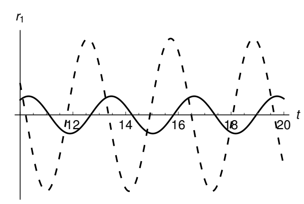
where $f(t)$ is the rate at which the second reagent is being added to the reaction and k is a constant.

a. [5 points] Write down the second-order linear equation which has r_1 as its solution.

Solution: From the first equation in the system, we have $r_2 = \frac{1}{9}\,r_1' + \frac{1}{3}\,r_1$, so that $r_2' = \frac{1}{9}\,r_1'' + \frac{1}{3}\,r_1'$. Plugging into the second equation, we have

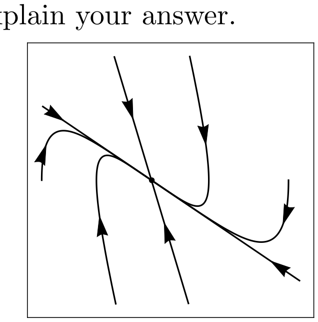
$$\frac{1}{9}\,r_1'' + \frac{1}{3}\,r_1' = k\,r_1 - \frac{1}{9}\,r_1' - \frac{1}{3}\,r_1 + f(t), \quad \text{or} \quad r_1'' + 4r_1' + (3 - 9k)r_1 = f(t).$$

b. [5 points] If $f(t) = \cos(\omega t)$ is the dashed curve in the figure below, for what values of k, if any, could the long-term behavior of r_1 be that shown by the solid curve? Explain your answer.



Solution: The solution for r_1 will be of the form $r_1 = r_c + r_p$, where $r_p = R\cos(\omega t - \delta)$. Thus, for this to be the long-term solution we must have $r_c \to 0$. This will be the case when the roots of the characteristic polynomial have negative real parts. The characteristic equation is $\lambda^2 + 4\lambda + (3 - 9k) = (\lambda + 2)^2 + (-1 - 9k) = 0$, for which $\lambda = -2 \pm \sqrt{1 + 9k}$. Thus we need $1 + 9k < 4$, or $k < 1/3$, for the real part of λ to be negative.

c. [5 points] If $f(t) = A_0$, a constant, for what values of k, if any, could the phase portrait for this system be similar to that shown in the figure below? Explain your answer.



Solution: Here we need the characteristic equation to have two negative, real roots. Thus we need $0 < 1 + 9k < 4$, so $-\frac{1}{9} < k < \frac{1}{3}$.