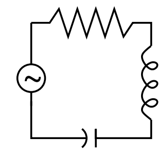
4. [13 points] Consider the RLC circuit shown to the right, below. This is modeled by $y'' + k\,y' + 2y = g(t)$, where $g(t)$ is the derivative of the input voltage and $0 < k < 2\sqrt{2}$ is proportional to the resistance of the resistor.

a. [9 points] If $g(t) = 4\cos(t)$, find the steady state response to the input. Write your answer in the form $R\cos(t - \alpha)$.



Solution: The steady state response will be the particular solution to the problem. Using the method of undetermined coefficients, let $y_p = A\cos t + B\sin t$. Then, plugging in and collecting terms in $\cos t$ and $\sin t$, we have

$$-A + Bk + 2A = 4, \quad \text{and} \quad -B - Ak + 2B = 0,.$$

These are $A + Bk = 4$, $-Ak + B = 0$. Solving by taking k times the first and adding, we have $(k^2+1)B = 4k$, so that $B = \frac{4k}{k^2+1}$. Then $A = \frac{4}{k^2+1}$. These are both positive, so in phase amplitude form we have

$$y_p = \sqrt{A^2 + B^2}\cos(t - \arctan(B/A)) = \frac{4}{\sqrt{k^2+1}}\cos(t - \arctan(k)).$$

b. [4 points] The amplitude of the steady state response to the forcing $g(t) = 4\cos(\omega t)$ is shown below, as a function of ω. What is the value of k in the equation? Why?

Solution: The expression for R above is for $\omega = 1$, so $\frac{4}{\sqrt{k^2+1}} = 2$, where we have read the value for R from the figure. This gives $\sqrt{k^2+1} = 2$, so $k^2 + 1 = 4$ and $k = \pm\sqrt{3}$. Given that it is an RLC circuit we can discard the negative value, taking $k = \sqrt{3}$.