6. [14 points] Find solutions to each of the following, as indicated.

a. [7 points] Find the general solution to $y'' + y' = \dfrac{1}{1+e^t}$. *(Hint:* $\int \frac{1}{1+e^t}\,dt = t - \ln(1+e^t)$.*)*

Solution: Solutions to the homogeneous problem are $y_1 = 1$ and $y_2 = e^{-t}$. We cannot here use the method of undetermined coefficients, and so use variation of parameters instead. Let $y_p = u_1 + u_2 e^{-t}$. Then

$$u_1' + u_2' e^{-t} = 0 \quad \text{and} \quad -u_2' e^{-t} = \frac{1}{1+e^t}.$$

The second equation gives $u_2' = -e^t(1+e^t)^{-1}$, so that $u_2 = -\ln(1+e^t)$. The first then gives $u_1' = -u_2' e^{-t} = 1/(1+e^t)$. Following the hint, $u_1 = t - \ln(1+e^t)$. Thus the general solution is

$$y = c_1 + c_2 e^{-t} + (t - \ln(1+e^t)) - e^{-t}\ln(1+e^t).$$

b. [7 points] Find the solution to $y'' - y' = u_2(t) - u_3(t)$, $y(0) = 0$, $y'(0) = 0$

Solution: We use Laplace transforms: transforming both sides of the equation, we have

$$s^2 Y - sY = \frac{1}{s}(e^{-2s} - e^{-3s}), \quad \text{so} \quad Y = \frac{1}{s^2(s-1)}(e^{-2s} - e^{-3s}).$$

To find the inverse transform of this, we first apply partial fractions to the rational function. Letting $\frac{1}{s^2(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1}$, we have after clearing the denominator

$$1 = As(s-1) + B(s-1) + Cs^2.$$

Setting $s = 0$, we have $B = -1$. If $s = 1$ we get $C = 1$. Finally, with $s = -1$, $1 = 2A + 2 + 1$, so that $A = -1$. Thus

$$\begin{aligned} \mathcal{L}^{-1}\{\frac{1}{s^2(s-1)}\} &= -\mathcal{L}^{-1}\{\frac{1}{s}\} - \mathcal{L}^{-1}\{\frac{1}{s^2}\} + \mathcal{L}^{-1}\{\frac{1}{s-1}\} \\ &= -1 - t + e^t = f(t). \end{aligned}$$

Thus, including the exponentials to get the expected step functions, we have

$$y = \mathcal{L}^{-1}\{Y\} = f(t-2)\,u_2(t) - f(t-3)\,u_3(t).$$