7. [14 points] Recall the linearized version of our laser model,

$$u' = -\gamma(Au + v)$$
$$v' = (A-1)\,u.$$

Consider the case of constant forcing (A = a constant) and the initial conditions $u(0) = u_0$, $v(0) = 0$.

a. [6 points] Find a system of algebraic equations for $U(s) = \mathcal{L}\{u(t)\}$ and $V(s) = \mathcal{L}\{v(t)\}$.

Solution: Transforming both sides of the system, we have

$$sU - u_0 = -\gamma A\,U - \gamma V$$
$$sV - 0 = (A-1)\,U,$$

or

$$(s + \gamma A)U + \gamma V = u_0$$
$$-(A-1)U + sV = 0.$$

b. [4 points] Solve your system from **(a)** to find $U(s)$ and $V(s)$.
(If you are unable to solve (a), consider the system $(s+a)U+bV = u_0$, $-cU+(s+a)V = 0$.*)*

Solution: Note that we have $V = \frac{A-1}{s}\,U$ from the second equation. Plugging this into the first, we have

$$(s+\gamma A)\,U + \frac{\gamma(A-1)}{s}\,U = \frac{s^2 + \gamma As + \gamma(A-1)}{s}U = u_0,$$

so that $U = \frac{s\,u_0}{s^2+\gamma As+\gamma(A-1)}$. Plugging this in for V, we have $V = \frac{(A-1)\,u_0}{s^2+\gamma As+\gamma(A-1)}$. With the alternate system, we have $U = \frac{s+a}{c}\,V$, so that $((s+a)^2 + bc)V = u_0 c$, or $V = \frac{u_0 c}{(s+a)^2+bc}$, and $U = \frac{u_0(s+a)}{(s+a)^2+bc}$.

c. [4 points] Recall that in the lab we solved the characteristic equation $\lambda^2+\gamma A\lambda+\gamma(A-1) = 0$, finding $\lambda^2 + \gamma A\lambda + \gamma(A-1) = (\lambda + \frac{1}{2}\,\lambda A)^2 + (-\frac{1}{4}\gamma^2A^2 + \gamma(A-1)) = (\lambda+\mu)^2 + \nu^2 = 0$ (so that $\lambda = -\mu \pm i\nu$). Use this to rewrite your solutions in **(b)** in terms of μ and ν. Find $u(t) = \mathcal{L}^{-1}\{U(s)\}$ and $v(t) = \mathcal{L}^{-1}\{V(s)\}$ in terms of μ and ν.
(If you are stuck, assume the denominator of your U *and* V *is of the form* $(s+a)^2 + b^2$.*)*

Solution: We have

$$U = \frac{s\,u_0}{(s+\mu)^2+\nu^2} = \frac{(s+\mu)\,u_0}{(s+\mu)^2+\nu^2} - \frac{\mu\,u_0}{(s+\mu)^2+\nu^2}, \qquad V = \frac{(A-1)\,u_0}{(s+\mu)^2+\nu^2}.$$

Inverting these, we have

$$u(t) = u_0\,e^{-\mu t}\cos(\nu t) - \frac{u_0}{\nu}\,e^{-\mu t}\sin(\nu t), \qquad v(t) = \frac{(A-1)}{\nu}\,u_0\,e^{-\mu t}\sin(\nu t).$$