7. [14 points] Recall the linearized version of our laser model,
\[
\begin{align*}
u' &= -\gamma(Au + v) \\
v' &= (A - 1)u.
\end{align*}
\]
Consider the case of constant forcing \((A = \text{a constant})\) and the initial conditions \(u(0) = u_0, v(0) = 0\).

a. [6 points] Find a system of algebraic equations for \(U\).

**Solution:** Transferring both sides of the system, we have
\[
\begin{align*}
sU - u_0 &= -\gamma AU - \gamma V \\
sV - 0 &= (A - 1) U,
\end{align*}
\]
or
\[
\begin{align*}
(s + \gamma A)U + \gamma V &= u_0 \\
-(A - 1)U + sV &= 0.
\end{align*}
\]

b. [4 points] Solve your system from (a) to find \(U(s)\) and \(V(s)\).

(If you are unable to solve (a), consider the system \((s+a)U + (s+a)V = u_0, -cU + (s+a)V = 0\).)

**Solution:** Note that we have \(V = \frac{A-1}{s}U\) from the second equation. Plugging this into the first, we have
\[
\frac{(s + \gamma A)U}{s} + \frac{\gamma(A - 1)}{s} U = \frac{s^2 + \gamma As + \gamma(A - 1)}{s} U = u_0,
\]
so that \(U = \frac{s u_0}{s^2 + \gamma As + \gamma(A - 1)}\). Plugging this in for \(V\), we have \(V = \frac{(A-1)u_0}{s^2 + \gamma As + \gamma(A - 1)}\).

With the alternate system, we have \(U = \frac{s+a}{c} V\), so that \(((s + a)^2 + bc)V = u_0 c\), or \(V = \frac{u_0(s+a)}{(s+a)^2 + bc}\), and \(U = \frac{u_0(s+a)}{(s+a)^2 + bc}\).

c. [4 points] Recall that in the lab we solved the characteristic equation \(\lambda^2 + \gamma A\lambda + \gamma(A - 1) = 0\), finding \(\lambda^2 + \gamma A\lambda + \gamma(A - 1) = (\lambda + \frac{1}{2} \lambda A)^2 + (-\frac{1}{4} \gamma^2 A^2 + \gamma(A - 1)) = (\lambda + \mu)^2 + \nu^2 = 0\) (so that \(\lambda = -\mu \pm iv\)). Use this to rewrite your solutions in (b) in terms of \(\mu\) and \(\nu\). Find \(u(t) = \mathcal{L}^{-1}\{U(s)\}\) and \(v(t) = \mathcal{L}^{-1}\{V(s)\}\) in terms of \(\mu\) and \(\nu\).

(If you are stuck, assume the denominator of your \(U\) and \(V\) is of the form \((s + a)^2 + b^2\).)

**Solution:** We have
\[
\begin{align*}
U &= \frac{s u_0}{(s + \mu)^2 + \nu^2} = \frac{(s + \mu) u_0}{(s + \mu)^2 + \nu^2} - \frac{\mu u_0}{(s + \mu)^2 + \nu^2}, \\
V &= \frac{(A - 1) u_0}{(s + \mu)^2 + \nu^2}.
\end{align*}
\]
Inverting these, we have
\[
\begin{align*}
u(t) &= u_0 e^{-\mu t} \cos(\nu t) - \frac{u_0}{\nu} e^{-\mu t} \sin(\nu t), \\
v(t) &= \frac{(A - 1)}{\nu} u_0 e^{-\mu t} \sin(\nu t).
\end{align*}
\]