1. [15 points] For each of the following, find explicit real-valued solutions as indicated.

a. [7 points] Find the solution to the initial value problem $y'' + 2y' + 17y = 0$, $y(0) = 1$, $y'(0) = 0$.

Solution: The solution will look like $y = e^{\lambda t}$. Plugging in, we have the characteristic equation $\lambda^2 + 2\lambda + 17 = (\lambda + 1)^2 + 16 = 0$. Thus $\lambda = -1 \pm 4i$. A general solution is $y = c_1 e^{-t}\cos(4t) + c_2 e^{-t}\sin(4t)$. Applying the initial conditions, we have $c_1 = 1$, and $-1 + 4c_2 = 0$, so that $c_2 = \frac{1}{4}$. Thus the solution is

$$y = e^{-t}\cos(4t) + \frac{1}{4}\,e^{-t}\sin(4t).$$

We can, of course, also solve this with Laplace transforms. The forward transform gives, with $Y = \mathcal{L}\{y\}$, $s^2Y - s + 2sY + 17Y = 0$, so that $Y = \frac{s}{s^2+2s+17} = \frac{(s+1)-1}{(s+1)^2+16} = \frac{s+1}{(s+1)^2+16} - \frac{1}{(s+1)^2+16}$. The two terms invert using the rules for cosine, sine, and $F(s - c)$ to give the result above.

b. [8 points] Find the general solution to the equation $y'' + 5y' + 4y = e^t + 8t$.

Solution: The complementary homogeneous solution will be a linear combination of exponentials $y = e^{\lambda t}$. Plugging in, we have $\lambda^2 + 5\lambda + 4 = (\lambda + 4)(\lambda + 1) = 0$, so $\lambda = -4$ or $\lambda = -1$ and $y_c = c_1 e^{-4t} + c_2 e^{-t}$.

To find the particular solution, consider the exponential and linear term separately. For the former, we guess $y_{p1} = ae^t$, so that $(1 + 5 + 4)ae^t = 10e^t$, and $a = \frac{1}{10}$. For the latter, we guess $y_{p2} = a_0 + a_1 t$, so that $5a_1 + 4a_0 + 4a_1 t = 8t$, and $a_1 = 2$, $a_0 = -\frac{5}{2}$.

Thus the general solution is

$$y = c_1 e^{-4t} + c_2 e^{-t} + \frac{1}{10}\,e^t - \frac{5}{2} + 2t.$$