2. [12 points] The following problems consider a non-homogeneous second-order linear differential equation $L[y] = g(t)$. Suppose that y_1 and y_2 are solutions to this equation, and that y_3, y_4, and y_5 are solutions to the complementary homogeneous problem $L[y] = 0$.

a. [3 points] Can you say what problem each of the following solve? If so, indicate what it is; if not, write "none." (No explanation necessary.)

i. $y_1 - y_2$

ii. $y_1 - y_3$

iii. $y_1 + y_2 + y_3 + y_4 + y_5$

Solution: i. $L[y_1 - y_2] = 0$; ii. $L[y_1 - y_3] = g(t)$; iii. $L[y_1 + y_2 + y_3 + y_4 + y_5] = 2g(t)$.

b. [3 points] Explain how you are able to determine your answers in (a), or why it is not possible to tell.

Solution: All of these stem from the linearity of the operator L, which means that $L[ay + bz] = aL[y] + bL[z]$. Thus we have:
i. $L[y_1 - y_2] = L[y_1] - L[y_2] = g(t) - g(t) = 0$,
ii. $L[y_1 - y_3] = L[y_1] - L[y_3] = g(t) - 0 = g(t)$, and
iii. $L[y_1+y_2+y_3+y_4+y_5] = L[y_1]+L[y_2]+L[y_3]+L[y_4]+L[y_5] = g(t)+g(t)+0+0+0 = 2g(t)$.

c. [6 points] The following statements are not guaranteed to be true. Explain why.

i. The solution to the initial problem $L[y] = 0$, $y(0) = y_0$, $y'(0) = v_0$ (for any y_0 and v_0) can be written as $y = c_1y_3 + c_2y_4 + c_3y_5$ for some c_1, c_2, and/or c_3 (where one or more of c_1, c_2, and c_3 may be zero).

ii. Because both y_1 and y_2 satisfy $L[y] = g(t)$, we must have $y_1 = y_2$.

Solution: i. This is false because we don't know that there are two linearly independent solutions from the three given. In order to be able to solve the initial value problem for any initial conditions we must start with a general solution, which requires that we have linearly independent solutions. The problem doesn't exclude the possibility that $y_3 = y_4 = y_5 = 0$, for example.
ii. This is false because y_1 and y_2 are only specified up to an additive multiple of a homogeneous solution: for example, y_2 could be y_1+y_3; then $L[y_1+y_3] = L[y_1]+L[y_3] = g(t) + 0$, but if $y_3 \neq 0$ the functions y_1 and y_2 are not equal.