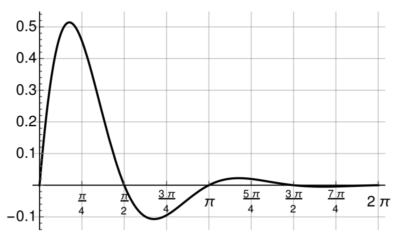
3. [15 points] For all of the following, the equations are linear, constant-coefficient, and second-order, with the coefficient of y'' picked to be one.

a. [5 points] If the differential equation is nonhomogeneous and the general solution is $y = c_1e^{-2t} + c_2e^{-3t} + 4\cos(2t)$, what is the differential equation?

Solution: The characteristic equation of the homogeneous problem must have roots -2 and -3, so it is $(\lambda + 2)(\lambda + 3) = \lambda^2 + 5\lambda + 6$. The differential equation is therefore $y'' + 5y' + 6y = g(t)$. The particular solution is $y_p = 4\cos(2t)$; plugging this in, we have $-16\cos(2t) - 10\sin(2t) + 24\cos(2t) = g(t)$, so $g(t) = 8\cos(2t) - 40\sin(2t)$, and the equation is

$$y'' + 5y' + 6y = 8\cos(2t) - 40\sin(2t)).$$

b. [5 points] If the graph to the right shows the movement of a unit mass on a spring with damping constant 2, set in motion with an initial velocity of 1 m/s, write an initial value problem modeling the position of the mass.



Solution: From the graph we see that $y(0) = 0$ and from the problem statement we know that $y'(0) = 1$. The solution is in the form of a decaying exponential, so the homogeneous solutions to the problem are of the form $e^{-at}\cos(bt)$ and $e^{-at}\sin(bt)$. The period of the oscillation is π, so $b = 2$. Then we know that $y'' + 2y' + ky = 0$, so that the characteristic equation is $\lambda^2 + 2\lambda + k = (\lambda + 1)^2 + k - 1 = 0$, which has roots $\lambda = -1 \pm i\sqrt{k-1}$. From the form of the solutions, we know $\sqrt{k-1} = 2$, so $k = 5$. Thus the initial value problem is

$$y'' + 2y' + 5y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

c. [5 points] Consider the (linear, constant-coefficient. . .) equation $L[y] = 0$ and the equivalent system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. If one solution to the equation $L[y] = 0$ is $y = e^{-t}$, what is a corresponding solution to the system? If the coefficient of y in the equation is 3, what is the differential equation?

Solution: If $y = e^{-t}$ is a solution to $L[y] = 0$, then $\mathbf{x} = \begin{pmatrix} e^{-t} & -e^{-t}\end{pmatrix}^T$ is a solution to the system, so that $\mathbf{v} = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda = -1$. Because the system is derived from a second order equation, and given the coefficient of y is 3, we know that $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -3 & k \end{pmatrix}$. Then, using $\mathbf{v}$ and $\lambda = -1$, we have $\begin{pmatrix} 1 & 1 \\ -3 & k+1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \mathbf{0}$, so $k = -4$ and the equation is

$$y'' + 4y' + 3y = 0.$$

Alternately, we know $y'' + ay' + 3y = 0$. Thus $\lambda^2 + a\lambda + 3 = (\lambda + 1)(\lambda + r) = 0$, for some r. Expanding the right-hand side and matching powers of λ, $r + 1 = a$ and $r = 3$. Thus $a = 4$, and we obtain the equation above.