3. [15 points] For all of the following, the equations are linear, constant-coefficient, and second-order, with the coefficient of \( y'' \) picked to be one.

a. [5 points] If the differential equation is nonhomogeneous and the general solution is \( y = c_1 e^{-2t} + c_2 e^{-3t} + 4 \cos(2t) \), what is the differential equation?

**Solution:** The characteristic equation of the homogeneous problem must have roots \(-2\) and \(-3\), so it is \((\lambda + 2)(\lambda + 3) = \lambda^2 + 5\lambda + 6\). The differential equation is therefore \( y'' + 5y' + 6y = g(t) \). The particular solution is \( y_p = 4 \cos(2t) \); plugging this in, we have \(-16 \cos(2t) - 10 \sin(2t) + 24 \cos(2t) = g(t) \), so \( g(t) = 8 \cos(2t) - 40 \sin(2t) \), and the equation is \( y'' + 5y' + 6y = 8 \cos(2t) - 40 \sin(2t) \).

b. [5 points] If the graph to the right shows the movement of a unit mass on a spring with damping constant 2, set in motion with an initial velocity of 1 m/s, write an initial value problem modeling the position of the mass.

**Solution:** From the graph we see that \( y(0) = 0 \) and from the problem statement we know that \( y'(0) = 1 \). The solution is in the form of a decaying exponential, so the homogeneous solutions to the problem are of the form \( e^{-at} \cos(bt) \) and \( e^{-at} \sin(bt) \). The period of the oscillation is \( \pi \), so \( b = 2 \). Then we know that \( y'' + 2y' + ky = 0 \), so that the characteristic equation is \( \lambda^2 + 2\lambda + k = (\lambda + 1)^2 + k - 1 = 0 \), which has roots \( \lambda = -1 \pm i\sqrt{k-1} \). From the form of the solutions, we know \( \sqrt{k-1} = 2 \), so \( k = 5 \). Thus the initial value problem is \( y'' + 2y' + 5y = 0 \), \( y(0) = 0 \), \( y'(0) = 1 \).

c. [5 points] Consider the (linear, constant-coefficient...) equation \( L[y] = 0 \) and the equivalent system \( \mathbf{x}' = A\mathbf{x} \). If one solution to the equation \( L[y] = 0 \) is \( y = e^{-t} \), what is a corresponding solution to the system? If the coefficient of \( y \) in the equation is 3, what is the differential equation?

**Solution:** If \( y = e^{-t} \) is a solution to \( L[y] = 0 \), then \( \mathbf{x} = (e^{-t} \quad -e^{-t})^T \) is a solution to the system, so that \( \mathbf{v} = (1 \quad -1)^T \) is an eigenvector of \( A \) with eigenvalue \( \lambda = -1 \). Because the system is derived from a second order equation, and given the coefficient of \( y \) is 3, we know that \( A = \begin{pmatrix} 0 & 1 \\ -3 & k \end{pmatrix} \). Then, using \( \mathbf{v} \) and \( \lambda = -1 \), we have \( \begin{pmatrix} 1 & 1 \\ -3 & k + 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \), so \( k = -4 \) and the equation is \( y'' + 4y' + 3y = 0 \).

Alternately, we know \( y'' + ay' + 3y = 0 \). Thus \( \lambda^2 + a\lambda + 3 = (\lambda + 1)(\lambda + r) = 0 \), for some \( r \). Expanding the right-hand side and matching powers of \( \lambda \), \( r + 1 = a \) and \( r = 3 \). Thus \( a = 4 \), and we obtain the equation above.