4. [14 points] Recall that the nonlinear model for the number of photons P and population inversion N in a ruby laser that we considered in lab 3 had an equilibrium point $(P, N) = (1, A-1)$. If we assume that $A = A_0 + a\cos(\omega t)$, with a a very small value, the dynamics of the system near the equilibrium point are modeled by the linear system $u' = -\gamma(Au + v) + \gamma a \cos(\omega t)$, $v' = (A-1)u$

a. [4 points] Rewrite this linear system as a second order equation in v.

Solution: Note that we have $u = \frac{1}{A-1} v'$ Plugging this into the first equation, we get $\frac{1}{A-1} v'' + \frac{\gamma A}{A-1} v' + \gamma v = \gamma a \cos(\omega t)$. Multiplying through by $A-1$, we obtain

$$v'' + \gamma A v' + \gamma(A-1)v = \gamma(A-1)a\cos(\omega t).$$

b. [6 points] Suppose that the second order equation you obtain in (a) is $v''+2v'+v = \cos(\omega t)$, so that the solution to the complementary homogeneous problem is $v_c = c_1 e^{-t} + c_2 t e^{-t}$. Set up the solution for v_p using variation of parameters, and solve them to obtain explicit equations for u_1' and u_2' in terms of t only. *(Do not solve these to find u_1 and u_2.)*

Solution: Our guess for v_p is $v_p = u_1 e^{-t} + u_2 t e^{-t}$. We know with variation of parameters, the equations that u_1 and u_2 must satisfy are $u_1' e^{-t} + u_2' t e^{-t} = 0$, $-u_1' e^{-t} + u_2'(e^{-t} - t e^{-t}) = \cos(\omega t)$. The first give $u_1' = -t u_2'$, so the second becomes $u_2' = e^t \cos(\omega t)$. Therefore, the equation for u_1 becomes $u_1' = -t e^t \cos(\omega t)$.

c. [4 points] It turns out that, for some A and B, $v_p = A\cos(\omega t) + B\sin(\omega t)$. Representative values of A and B are given for different values of ω in the table below. Does the system exhibit resonance? Write the response v_p to a forcing of $\cos(2t)$ in phase-amplitude form.

$\omega =$	1	2	3	4	5
$A =$	0	-0.12	-0.08	-0.06	-0.04
$B =$	1	0.64	0.36	0.22	0.15

Solution: We see that the amplitude of v_p, $R = \sqrt{A^2 + B^2}$, is never larger than one, so there is no resonance. When $\omega = 2$, $A = -0.12$ and $B = 0.64$, so $R = \sqrt{0.144 + 0.4096} \approx \sqrt{0.424}$. The phase shift δ has $A < 0$ and $B > 0$, so we need $\delta = \pi - \arctan(0.64/0.12) \approx \pi - \arctan(5.33)$. Thus $v_p = \sqrt{0.424}\cos(2t - (\pi - \arctan(5.33)))$.