5. [14 points] Consider the initial value problem $y'' + 4y' + 4y = 9e^{-2t}$, $y(0) = 1$, $y'(0) = 0$.

a. [6 points] Solve the problem *without* using Laplace transforms.

Solution: We first note that with $y = e^{\lambda t}$, the characteristic equation for the complementary homogeneous solution is $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$. Thus $\lambda = -2$, repeated, and $y_c = c_1 e^{-2t} + c_2 te^{-2t}$.
For the particular solution, we therefore guess $y_p = at^2t^{-2t}$, so that $y_p' = 2ate^{-2t} - 2at^2e^{-2t}$, and $y_p'' = 2ae^{-2t} - 8ate^{-2t} + 4at^2e^{-2t}$. Plugging in, we have

$$2ae^{-2t} - 8ate^{-2t} + 4at^2e^{-2t} + 4(2ate^{-2t} - 2at^2e^{-2t}) + 4at^2e^{-2t} = 9e^{-2t},$$

so that $a = \frac{9}{2}$. Thus the general solution is

$$y = c_1e^{-2t} + c_2te^{-2t} + \frac{9}{2}t^2e^{-2t}.$$

Applying the initial conditions, $y(0) = c_1 = 1$, and $y'(0) = -2c_1 + c_2 = -2 + c_2 = 0$, so that $c_2 = 2$. The solution is therefore

$$y = e^{-2t} + 2te^{-2t} + \frac{9}{2}t^2e^{-2t}.$$

b. [8 points] Solve the problem *with* Laplace transforms.

Solution: We transform both sides of the equation to find, with $Y = \mathcal{L}\{y\}$,

$$s^2Y - s + 4sY - 4 + 4Y = \frac{9}{s+2},$$

so that $Y = \frac{s+4}{(s+2)^2} + \frac{9}{(s+2)^3}$. The easiest way to find the inverse transform is to rewrite Y solely in terms of $s + 2$:

$$Y = \frac{(s+2)+2}{(s+2)^2} + \frac{9}{(s+2)^3} = \frac{1}{s+2} + \frac{2}{(s+2)^2} + \frac{9}{(s+2)^3},$$

so that (applying for the second and third the rules for $F(s - c)$ and $1/s^n$)

$$y = e^{-2t} + 2te^{-2t} + \frac{9}{2}t^2e^{-2t}.$$

We can also find the inverse transform of the first by using partial fractions and letting $\frac{s+4}{(s+2)^2} = \frac{A_0}{s+2} + \frac{A_1}{(s+2)^2}$. Clearing the denominator, $s+4 = A_0(s+2) + A_1$, so that $A_0 = 1$ and $A_1 = 2$. Thus

$$Y = \frac{1}{s+2} + \frac{2}{(s+2)^2} + \frac{9}{(s+2)^3},$$

and the inverse transform is as before.