6. [15 points] Complete each of the following problems having to do with the Laplace transform.

a. [5 points] Find the inverse Laplace transform of $F(s) = \dfrac{5s}{s^2+4s+6}$

Solution: Note that $s^2+4s+6 = (s+2)^2+2$. Thus $\frac{5s}{s^2+4s+6} = \frac{5(s+2)}{(s+2)^2+2} - \frac{10}{(s+2)^2+2}$, and the inverse transform is

$$\mathcal{L}^{-1}\{\frac{5(s+2)}{(s+2)^2+2} - \frac{10}{(s+2)^2+2}\} = 5e^{-2t}\cos(\sqrt{2}\,t) - 5\sqrt{2}\,e^{-2t}\sin(\sqrt{2}\,t).$$

b. [5 points] Given that $F(s) = \mathcal{L}\{f(t)\}$, use the integral definition of the Laplace transform to derive the transform rule $-F'(s) = \mathcal{L}\{tf(t)\}$.

Solution: We have $F(s) = \int_0^\infty e^{-st} f(t)\,dt$. Thus, assuming that the integral converges in a manner that allows differentiation through the integral sign,

$$\begin{aligned} -F'(s) = -\int_0^\infty \frac{d}{ds}(e^{-st})f(t)\,dt &= -\int_0^\infty -te^{-st}f(t)\,dt \\ &= \int_0^\infty e^{-st}tf(t)\,dt = \mathcal{L}\{tf(t)\}. \end{aligned}$$

c. [5 points] Consider the initial value problem $ty''+y=0$, $y(0)=1$, $y'(0)=0$. If $Y=\mathcal{L}\{y\}$, what equation does Y satisfy?

Solution: Note that $\mathcal{L}\{y''\} = s^2Y - s$, and that the transform rule in part (b) above (also, rule B on the formula sheet) gives $\mathcal{L}\{tf(t)\} = -F'(s)$. Thus, with $f(t) = y''$ and $F(s) = s^2Y(s) - 1$, we have $\mathcal{L}\{ty''\} = -s^2Y'(s) - 2sY(s) + 1$. Transforming the differential equation, we therefore have

$$-s^2Y'(s) - 2sY(s) + 1 + Y(s) = 0,$$

a first-order equation for $Y(s)$. Note that we can rewrite this as $Y' + \frac{2s-1}{s^2}Y = \frac{1}{s^2}$, so the equation is first-order and linear, and therefore solvable at least in principle. It is not clear if we would be able to invert the resulting $Y(s)$, however.