6. [15 points] Complete each of the following problems having to do with the Laplace transform.

a. [5 points] Find the inverse Laplace transform of \( F(s) = \frac{5s}{s^2 + 4s + 6} \).

**Solution:** Note that \( s^2 + 4s + 6 = (s + 2)^2 + 2 \). Thus \( \frac{5s}{s^2 + 4s + 6} = \frac{5(s + 2)}{(s + 2)^2 + 2} - \frac{10}{(s + 2)^2 + 2} \), and the inverse transform is

\[
\mathcal{L}^{-1}\left\{ \frac{5(s + 2)}{(s + 2)^2 + 2} - \frac{10}{(s + 2)^2 + 2} \right\} = 5e^{-2t}\cos(\sqrt{2}t) - 5\sqrt{2}e^{-2t}\sin(\sqrt{2}t).
\]

b. [5 points] Given that \( F(s) = \mathcal{L}\{f(t)\} \), use the integral definition of the Laplace transform to derive the transform rule \( -F'(s) = \mathcal{L}\{tf(t)\} \).

**Solution:** We have \( F(s) = \int_0^\infty e^{-st}f(t)\,dt \). Thus, assuming that the integral converges in a manner that allows differentiation through the integral sign,

\[
-F'(s) = -\int_0^\infty \frac{d}{ds}(e^{-st})f(t)\,dt = -\int_0^\infty -te^{-st}f(t)\,dt = \int_0^\infty e^{-st}tf(t)\,dt = \mathcal{L}\{tf(t)\}.
\]

c. [5 points] Consider the initial value problem \( ty'' + y = 0 \), \( y(0) = 1 \), \( y'(0) = 0 \). If \( Y = \mathcal{L}\{y\} \), what equation does \( Y \) satisfy?

**Solution:** Note that \( \mathcal{L}\{y''\} = s^2Y - s \), and that the transform rule in part (b) above (also, rule B on the formula sheet) gives \( \mathcal{L}\{tf(t)\} = -F'(s) \). Thus, with \( f(t) = y'' \) and \( F(s) = s^2Y(s) - s \), we have \( \mathcal{L}\{ty''\} = -s^2Y'(s) - 2sY(s) + 1 \). Transforming the differential equation, we therefore have

\[
-s^2Y'(s) - 2sY(s) + 1 + Y(s) = 0,
\]
a first-order equation for \( Y(s) \). Note that we can rewrite this as \( Y' + \frac{2s-1}{s^2}Y = \frac{1}{s^2} \), so the equation is first-order and linear, and therefore solvable at least in principle. It is not clear if we would be able to invert the resulting \( Y(s) \), however.