7. [15 points] Consider the system of differential equations $x' = 3x + 4y$, $y' = 2x + y$, with initial conditions $x(0) = 0$, $y(0) = 2$.

a. [6 points] Using Laplace transforms, find explicit equations for $X = \mathcal{L}\{x\}$ and $Y = \mathcal{L}\{y\}$.

Solution: We have $X = \mathcal{L}\{x\}$ and $Y = \mathcal{L}\{y\}$. Then, transforming both equations, we have $sX = 3X + 4Y$ and $sY - 2 = 2X + Y$. Thus $Y = \frac{1}{4}(s-3)X$, and the second equation becomes $(s-1)\frac{1}{4}(s-3)X - 2X = 2$, or $X = \frac{8}{s^2-4s-5} = \frac{8}{(s-5)(s+1)}$. Plugging back into the equation for Y, we have

$$X = \frac{8}{(s-5)(s+1)}, \quad Y = \frac{2(s-3)}{(s-5)(s+1)}.$$

b. [4 points] Find x and y in terms of any constants you may have in partial fractions expansions of X and Y (that is, do not solve for the values of those constants).

Solution: We can find the inverse transforms for both of these by using partial fractions: for X, we have $X = A/(s-5) + B/(s+1)$. For Y, we have $Y = C/(s-5) + D/(s+1)$. Thus

$$x = Ae^{5t} + Be^{-t}, \quad y = Ce^{5t} + De^{-t}.$$

If we complete the solution, which is not required in the problem, we have in the partial fractions decomposition for X $8 = A(s+1) + B(s-5)$. Plugging in $s = 5$ and $s = -1$, we get $A = \frac{4}{3}$ and $B = -\frac{4}{3}$. Similarly for Y, we have $2s - 6 = C(s+1) + D(s-5)$. Plugging in $s = 5$ and $s = -1$, we have $C = \frac{2}{3}$ and $D = \frac{4}{3}$. Thus the wolution is

$$x = \frac{4}{3}e^{5t} - \frac{4}{3}e^{-t}, \quad y = \frac{2}{3}e^{5t} + \frac{4}{3}e^{-t}.$$

c. [5 points] If we rewrote the system as a second order differential equation $L[y] = 0$ for y, what would the characteristic equation for λ be? What is the linear operator L?

Solution: The characteristic equation for λ can be seen from the denominator of X and Y, or from the exponentials in x and y: $(\lambda - 5)(\lambda + 1) = \lambda^2 - 4\lambda - 5 = 0$. Thus the operator L is $L = D^2 - 4D - 5$, or some constant multiple thereof.