1. [12 points] Find each of the following.

a. [7 points] Use the integral definition of the Laplace transform to find $F(s) = \mathcal{L}\{f(t)\}$, where

$$f(t) = \begin{cases} e^{-1}, & 0 < t \leq 1 \\ e^{-t}, & 1 < t < \infty. \end{cases}$$

Solution: Applying the transform, we have

$$\begin{aligned} F(s) &= \int_0^\infty f(t)e^{-st}\,dt \\ &= \int_0^1 e^{-1}e^{-st}\,dt + \int_1^\infty e^{-(s+1)t}\,dt \\ &= -\frac{1}{s}\,e^{-1}e^{-st}\Big|_{t=0}^{t=1} - \frac{1}{s+1}\,e^{-(s+1)t}\Big|_{t=1}^{t\to\infty} \\ &= \frac{1}{s}(e^{-1} - e^{-(s+1)}) + \frac{1}{s+1}\,e^{-(s+1)}. \end{aligned}$$

b. [5 points] Give another function $g(t)$ for which $\mathcal{L}\{g(t)\} = F(s)$. Explain your answer briefly.

Solution: Because the transform is defined as an integral, any single point differences between g and f will not change the transform. Thus, we could take

$$g(t) = \begin{cases} e^{-1}, & 0 < t < 1 \\ 1, & t = 1 \\ e^{-t}, & 1 < t < \infty, \end{cases}$$

or any other variation where we make $g(t)$ piecewise continuous with isolated points that have a different value from $f(t)$.