2. [15 points] In each of the following $\mathcal{L}$ is the Laplace transform operator, and, in (b), L is a linear, constant-coefficient differential operator.

a. [5 points] If $x' = 3x + 4y$ and $y' = 2x - y$, with initial conditions $x(0) = 0$ and $y(0) = 2$, find $X = \mathcal{L}\{x\}$ and $Y = \mathcal{L}\{y\}$.

Solution: Taking the forward transform of both equations, we have $sX - 0 = 3X + 4Y$, $sY - 2 = 2X - Y$. Rewriting as a linear algebraic system, we have

$$(s-3)X - 4Y = 0$$
$$-2X + (s+1)Y = 2.$$

Taking $(s + 1)$ times the first and 4 times the second and adding, we find

$$X = \frac{8}{(s+1)(s-3) - 8} = \frac{8}{s^2 - 2s - 11}.$$

Similarly, taking 2 times the first and $(s - 3)$ times the second and adding, we get

$$Y = \frac{2(s-3)}{(s-3)(s+1) - 8} = \frac{2(s-3)}{s^2 - 2s - 11}.$$

b. [5 points] Suppose that when solving an equation $L[y] = f(t)$, $y(0) = y_0$, $y'(0) = v_0$ using the Laplace transform, we find

$$\mathcal{L}\{y(t)\} = Y(s) = \frac{5}{(s+1)(s+2)} + \frac{s}{(s+1)(s+2)(s^2+4)}.$$

What are L, $f(t)$, and the initial conditions y_0 and v_0?

Solution: The factors $(s + 1)(s + 2) = s^2 + 3s + 2$ in both terms of the denominator indicate that $L = D^2 + 3D + 2$. The second term is the response to the forcing term $f(t)$, for which we see $\mathcal{L}\{f(t)\} = \frac{s}{s^2+4}$, so $f(t) = \cos(2t)$. Finally, the first term on the right-hand side is the response to the initial forcing. We see that there is no s in the numerator, so $y_0 = 0$; then $v_0 = 5$ to give the desired form.

c. [5 points] Derive the transform rule $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ for a continuous function $f(t)$.

Solution: We apply the integral definition of the transform:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)\, e^{-st}\, dt.$$

Integrating by parts with $u = e^{-st}$ and $v' = f'(t)$, we have

$$\int_0^\infty f'(t)\, e^{-st}\, dt = e^{-st} f(t)\Big|_{t=0}^{t\to\infty} + s\int_0^\infty e^{-st} f(t)\, dt = -f(0) + s\mathcal{L}\{f(t)\},$$

assuming that the limit in the first term is well behaved as $t \to \infty$.