3. [16 points] For $t > 0$, consider the differential equation $L[y] = y'' - 3t^{-1}y' - 5t^{-2}y = 0$.

a. [4 points] Determine which of $y_1 = t^{-1}$, $y_2 = 1$, $y_3 = t$, $y_4 = \frac{1+t^6}{t}$, and $y_5 = t^5$ are solutions to $L[y] = 0$.

Solution: Note that $y_4 = t^{-1} + t^5 = y_1 + y_5$, so, because the operator is linear, y_4 will be a solution if y_1 and y_5 are. Plugging in, we see that

$$L[y_1] = 2t^{-3} + 3t^{-3} - 5t^{-3} = 0, \qquad L[y_2] = 0 - 0 - 5t^{-2} \neq 0$$
$$L[y_3] = 0 - 3t^{-1} - 5t^{-1} \neq 0, \quad \text{and} \qquad L[y_5] = 20t^3 - 15t^3 - 5t^3 = 0.$$

Thus y_1, y_4, and y_5 are solutions.

b. [4 points] Write a general solution to $L[y] = 0$. Explain why your solution is correct.

Solution: We need two linearly independent solutions to write a general solution, and $W[y_1, y_5] = \begin{vmatrix} t^{-1} & t^5 \\ -t^{-2} & 5t^4 \end{vmatrix} = 6t^3 \neq 0$ (for $t > 0$), so they are linearly independent. A general solution is

$$y = c_1 t^{-1} + c_2 t^5.$$

c. [4 points] If you were solving $L[y] = 5t^5$, what forms could the particular solution take (that is, what could you guess for y_p)? Why?

Solution: This is a non-constant coefficient problem, so we do not expect that the method of undetermined coefficients will work. (With some effort, we could come up with a polynomial guess that will work, but this doesn't follow the rules we have for the method.) Therefore, it makes best sense to use variation of parameters. The form of the solution is then $y_p = u_1(t)t^{-1} + u_2(t)t^5$.

d. [4 points] Find y_p.

Solution: We know that $t^{-1}u_1' + t^5 u_2' = 0$, and $-t^{-2}u_1' + 5t^4 u_2' = 5t^5$. The first gives $u_1' = -t^6 u_2'$, so, plugging into the second, $6t^4 u_2' = 5t^5$. This gives $u_2' = \frac{5}{6}t$, so $u_1' = -\frac{5}{6}t^7$. Integrating, $u_1 = -\frac{5}{48}t^8$ and $u_2 = \frac{5}{12}t^2$. Thus

$$y_p = -\frac{5}{48}\,t^8\,t^{-1} + \frac{5}{12}\,t^2\,t^5 = \frac{5}{16}\,t^7.$$

If we memorized the formula for variation of parameters, we have (recalling that $W[y_1, y_2] = 6t^3$)

$$\begin{aligned} y_p &= -t^{-1}\int \frac{(t^5)(5t^5)}{6t^3}\,dt + t^5 \int \frac{(t^{-1})(5t^5)}{6t^3}\,dt \\ &= -t^{-1}\int \frac{5}{6}\,t^7\,dt + t^5 \int \frac{5}{6}\,t\,dt = \frac{5}{16}\,t^7 \end{aligned}$$

(where we have used the fact that the two integrals are exactly those we just calculated above).