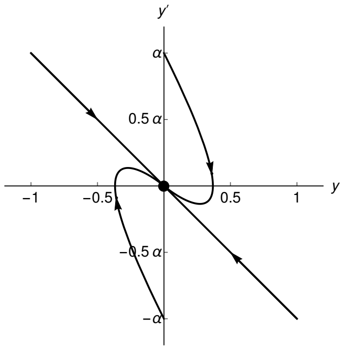
4. [15 points] Consider the homogeneous problem $L[y] = my'' + \gamma y' + ky = 0$.

a. [5 points] If this models critically damped harmonic motion, find the general solution to the problem.

Solution: If this is critically damped, there is a repeated real root to the characteristic equation $mr^2 + \gamma r + k = 0$. Thus, the characteristic equation is of the form $r^2 + 2\alpha r + \alpha^2$, so that $r = -\alpha$. The general solution is then given by $y = c_1 e^{-\alpha t} + c_2 t e^{-\alpha t}$. Relating this to the original equation, $\alpha = \frac{\gamma}{2m}$, and in this case $\alpha^2 = \frac{k}{m}$.

b. [5 points] Sketch a phase portrait for the system for the case when this represents critically damped harmonic motion.

Solution: In this case, the eigenvector for the system is $\mathbf{v} = \begin{pmatrix} 1 \\ -\alpha \end{pmatrix}$. Note that the second solution gives the vector form $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -\alpha \end{pmatrix} te^{-\alpha t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\alpha t}$. Thus there is a single straight line in the phase portrait, $y = -\alpha x$, along which solutions approach the origin. If we consider an initial condition $y(0) = 0$, $y'(0) = 1$, the trajectory will be described by the second solution, and will initially increase in the direction $\mathbf{v}$ and then collapse to the origin. This gives the phase portrait shown below.



c. [5 points] Suppose that we decrease γ in our equation very slightly from the critically damped case we considered in (a) and (b). Sketch the phase portrait for the new system. Why does it change as it does? What type of damping are we seeing now?

Solution: If γ decreases slightly, then solutions of the characteristic equation will become complex valued, $r = -\frac{\gamma}{2m} \pm i\frac{1}{2m}\sqrt{4mk - \gamma^2} = -\mu \pm i\nu$, and the general solution will be $y = c_1 e^{-\mu t}\cos(\nu t) + c_2 e^{-\mu t}\sin(\nu t)$. The phase portrait will then be spirals, with no straight line solutions, becoming something like the figure below. The system is in this case underdamped.