6. [15 points] For each of the following find explicit, real-valued solutions as indicated. For this problem do **not** use Laplace transforms. Note that you do not need to simplify numeric expressions.

a. [8 points] Find the solution to $3y'' + 4y' + y = 5e^{-t}$, $y(0) = y'(0) = 0$

Solution: We look for a homogeneous solution of the form $y = e^{rt}$, so that $3r^2+4r+1 = (3r+1)(r+1) = 0$, and $r = -\frac{1}{3}$ or $r = -1$. The complementary homogeneous solution is therefore $y = c_1e^{-t/3} + c_2e^{-t}$. For the particular solution we look for $y = Ate^{-t}$, because the exponential is part of the homogeneous solution. Plugging this into the equation, we have

$$3(-2Ae^{-t} + Ate^{-t} + 4(Ae^{-t} - Ate^{-t}) + Ate^{-t} = -2Ae^{-t} = 5e^{-t},$$

so that $A = -\frac{5}{2}$. Our general solution is therefore

$$y = c_1e^{-t/3} + c_2e^{-t} - \frac{5}{2}\,te^{-t}.$$

The initial condition requires that $c_1 + c_2 = 0$ and $-\frac{1}{3}c_1 - c_2 - \frac{5}{2} = 0$. Thus (adding), $c_1 = \frac{15}{4}$ and $c_2 = -\frac{15}{4}$, so that

$$y = \frac{15}{4}e^{-t/3} - \frac{15}{4}e^{-t} - \frac{5}{2}\,te^{-t}.$$

b. [7 points] Find the general solution to $y'' + 2y' + 10y = 5t$.

Solution: The homogeneous solution is given by $y = e^{rt}$, so that $r^2 + 2r + 10 = (r + 1)^2 + 9 = 0$, and $r = -1 \pm 3i$. The homogeneous solution is therefore $y = c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t)$. The particular solution is $y_p = At + B$, so that $A = \frac{1}{2}$ and $B = -\frac{1}{10}$, so that the general solution is

$$y = c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t) + \frac{1}{2}t - \frac{1}{10}.$$