7. [15 points] For each of the following find explicit, real-valued solutions as indicated. For this
problem, do use Laplace transforms.

a. [8 points] Solve \( y'' + 2y' + 10y = 5, \) \( y(0) = 1, \) \( y'(0) = 2. \)

**Solution:** With \( Y = \mathcal{L}\{y\}, \) the forward transform gives \( s^2Y - 2s + 2sY' - 2 + 10Y = \frac{5}{s}, \) so that

\[
Y = \frac{s + 4}{(s + 1)^2 + 9} + \frac{5}{s((s + 1)^2 + 9)}.
\]

To invert the second of these, we use partial fractions, letting \( \frac{A}{s} + \frac{B + C}{(s + 1)^2 + 9} = \frac{5}{s((s + 1)^2 + 9)}. \) Clearing the denominator, we must have \( A(s^2 + 2s + 10) + Bs^2 + Cs = 5. \) Thus \( A = \frac{1}{2}; \) matching terms in \( s^2, \) \( B = -A = -\frac{1}{2}; \) and matching terms in \( s, \) \( C = -2A = -1. \) We can therefore rewrite \( Y \) as

\[
Y = \frac{(s + 1) + 3}{(s + 1)^2 + 9} + \frac{1}{2s} - \frac{1}{2} \frac{(s + 1) + 1}{(s + 1)^2 + 9}.
\]

Inverting, we have

\[
y = e^{-t} \cos(3t) + e^{-t} \sin(3t) + \frac{1}{2} - \frac{1}{2} e^{-t} \cos(3t) - \frac{1}{6} e^{-t} \sin(3t),
\]
or, combining terms,

\[
y = \frac{1}{2} e^{-t} \cos(3t) + \frac{5}{6} e^{-t} \sin(3t) + \frac{1}{2}.
\]

b. [7 points] Find the solution to \( y'' + 5y' + 6y = e^{-3t}, \) \( y(0) = y'(0) = 0. \)

**Solution:** As before, the forward transform gives \( (s^2 + 5s + 6)Y = \frac{1}{s+3}, \) so that \( Y = \frac{1}{(s+2)(s+3)^2}. \) Using partial fractions,

\[
\frac{1}{(s+2)(s+3)^2} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{(s+3)^2}.
\]

Clearing denominators, \( A(s + 3)^2 + B(s + 2)(s + 3) + C(s + 2) = 1. \) Plugging in \( s = -2, \) \( A = 1; \) plugging in \( s = -3, \) \( C = -1. \) Finally, with \( s = -1, \) \( 4A + 2B + C = 3 + 2B = 1, \) and \( B = -1. \) Thus

\[
y = \mathcal{L}^{-1}\{\frac{1}{s+2} - \frac{1}{s+3} - \frac{1}{(s+3)^2}\} = e^{-2t} - e^{-3t} - te^{-3t}.
\]