7. [15 points] For each of the following find explicit, real-valued solutions as indicated. For this problem, **do** use Laplace transforms.

a. [8 points] Solve $y'' + 2y' + 10y = 5$, $y(0) = 1$, $y'(0) = 2$.

Solution: With $Y = \mathcal{L}\{y\}$, the forward transform gives $s^2Y - 2 - s + 2sY - 2 + 10Y = \frac{5}{s}$, so that

$$Y = \frac{s+4}{(s+1)^2+9} + \frac{5}{s((s+1)^2+9)}.$$

To invert the second of these, we use partial fractions, letting $\frac{A}{s} + \frac{Bs+C}{(s+1)^2+9} = \frac{5}{s((s+1)^2+9)}$. Clearing the denominator, we must have $A(s^2+2s+10) + Bs^2 + Cs = 5$. Thus $A = \frac{1}{2}$; matching terms in s^2, $B = -A = -\frac{1}{2}$; and matching terms in s, $C = -2A = -1$. We can therefore rewrite Y as

$$Y = \frac{(s+1)+3}{(s+1)^2+9} + \frac{1}{2s} - \frac{1}{2}\frac{(s+1)+1}{(s+1)^2+9}.$$

Inverting, we have

$$y = e^{-t}\cos(3t) + e^{-t}\sin(3t) + \frac{1}{2} - \frac{1}{2}e^{-t}\cos(3t) - \frac{1}{6}e^{-t}\sin(3t),$$

or, combining terms,

$$y = \frac{1}{2}e^{-t}\cos(3t) + \frac{5}{6}e^{-t}\sin(3t) + \frac{1}{2}.$$

b. [7 points] Find the solution to $y'' + 5y' + 6y = e^{-3t}$, $y(0) = y'(0) = 0$.

Solution: As before, the forward transform gives $(s^2+5s+6)Y = \frac{1}{s+3}$, so that $Y = \frac{1}{(s+2)(s+3)^2}$. Using partial fractions,

$$\frac{1}{(s+2)(s+3)^2} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{(s+3)^2}.$$

Clearing denominators, $A(s+3)^2 + B(s+2)(s+3) + C(s+2) = 1$. Plugging in $s = -2$, $A = 1$; plugging in $s = -3$, $C = -1$. Finally, with $s = -1$, $4A + 2B + C = 3 + 2B = 1$, and $B = -1$. Thus

$$y = \mathcal{L}^{-1}\{\frac{1}{s+2} - \frac{1}{s+3} - \frac{1}{(s+3)^2}\} = e^{-2t} - e^{-3t} - te^{-3t}.$$