1. [15 points] Find explicit, real-valued solutions to each of the following, as indicated. For this problem, **DO NOT use Laplace transforms**.

a. [8 points] Find the solution $z(t)$ to the initial value problem $3z'' + 12z' + 39z = 6e^{-t}$, $z(0) = \frac{1}{5}$, $z'(0) = 0$

Solution: This problem is nonhomogeneous, linear, and constant-coefficient. The general solution will be $z = z_c + z_p$, where z_c is the general solution to the complementary homogeneous problem. For this we look for a solution $z = e^{\lambda t}$. Plugging in to the homogeneous equation, we have $3\lambda^2 + 12\lambda + 39 = 3(\lambda^2 + 4\lambda + 13) = 3((\lambda + 2)^2 + 9) = 0$. Thus $\lambda = -2 \pm 3i$. Separating the real and imaginary parts of the resulting complex exponential, we have $z_c = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t)$.

Then, to find z_p, we use undertermined coefficients and look for a solution of the form $z_p = Ae^{-t}$. Plugging into the differential equation, we have $3A - 12A + 39A = 6$, or $10A = 2$, so that $A = \frac{1}{5}$. Thus

$$z = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t) + \frac{1}{5}\,e^{-t}.$$

Applying the initial conditions, we have $z(0) = c_1 + \frac{1}{5} = \frac{1}{5}$, so $c_1 = 0$. Then $z'(0) = 3c_2 - \frac{1}{5} = 0$, so that $c_2 = \frac{1}{15}$, and

$$z = \frac{1}{15}e^{-2t}\sin(3t) + \frac{1}{5}\,e^{-t}.$$

b. [7 points] Find the general solution $y(t)$ to $y'' + 5y' + 6y = \cos(t)$.

Solution: Again, the general solution will be $y = y_c + y_p$. For y_c, the characteristic equation is $\lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3) = 0$, so $\lambda = -2$ or $\lambda = -3$, and $y_c = c_1 e^{-2t} + c_2 e^{-3t}$.

For y_p, we again use the method of undetermined coefficients and guess $y_p = a\cos(t) + b\sin(t)$. Plugging into the equation, we have

$$(-a\cos(t) - b\sin(t)) + (-5a\sin(t) + 5b\cos(t)) + 6a\cos(t) + 6b\sin(t) = \cos(t).$$

Collecting the $\cos(t)$ and $\sin(t)$ terms, this is $5a + 5b = 1$ and $-5a + 5b = 0$. Adding the two we get $b = \frac{1}{10}$, so that from the second $a = \frac{1}{10}$, and the

$$y = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{10}\cos(t) + \frac{1}{10}\sin(t).$$