1. [15 points] Find explicit, real-valued solutions to each of the following, as indicated. For this problem, **DO NOT use Laplace transforms**.

a. [8 points] Find the solution \( z(t) \) to the initial value problem

\[
3z'' + 12z' + 39z = 6e^{-t}, \quad z(0) = \frac{1}{5}, \quad z'(0) = 0
\]

**Solution:** This problem is nonhomogeneous, linear, and constant-coefficient. The general solution will be \( z = z_c + z_p \), where \( z_c \) is the general solution to the complementary homogeneous problem. For this we look for a solution \( z = e^{\lambda t} \). Plugging in to the homogeneous equation, we have

\[
3\lambda^2 + 12\lambda + 39 = 3(\lambda^2 + 4\lambda + 13) = 3((\lambda + 2)^2 + 9) = 0.
\]

Thus \( \lambda = -2 \pm 3i \). Separating the real and imaginary parts of the resulting complex exponential, we have

\[
z_c = c_1 e^{-2t} \cos(3t) + c_2 e^{-2t} \sin(3t).
\]

Then, to find \( z_p \), we use undertermined coefficients and look for a solution of the form \( z_p = Ae^{-t} \). Plugging into the differential equation, we have

\[
10A = 2, \quad A = \frac{1}{5}.
\]

Thus \( z = c_1 e^{-2t} \cos(3t) + c_2 e^{-2t} \sin(3t) + \frac{1}{5} e^{-t} \).

Applying the initial conditions, we have \( z(0) = c_1 + \frac{1}{5} = \frac{1}{5} \), so \( c_1 = 0 \). Then \( z'(0) = 3c_2 - \frac{1}{5} = 0 \), so that \( c_2 = \frac{1}{15} \), and

\[
z = \frac{1}{15} e^{-2t} \sin(3t) + \frac{1}{5} e^{-t}.
\]

b. [7 points] Find the general solution \( y(t) \) to \( y'' + 5y' + 6y = \cos(t) \).

**Solution:** Again, the general solution will be \( y = y_c + y_p \). For \( y_c \), the characteristic equation is \( \lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3) = 0 \), so \( \lambda = -2 \) or \( \lambda = -3 \), and \( y_c = c_1 e^{-2t} + c_2 e^{-3t} \).

For \( y_p \), we again use the method of undetermined coefficients and guess \( y_p = a \cos(t) + b \sin(t) \). Plugging into the equation, we have

\[
(-a \cos(t) - b \sin(t)) + (-5a \sin(t) + 5b \cos(t)) + 6a \cos(t) + 6b \sin(t) = \cos(t).
\]

Collecting the \( \cos(t) \) and \( \sin(t) \) terms, this is \( 5a + 5b = 1 \) and \( -5a + 5b = 0 \). Adding the two we get \( b = \frac{1}{10} \), so that from the second \( a = \frac{1}{10} \), and the

\[
y = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{10} \cos(t) + \frac{1}{10} \sin(t).
\]