2. [15 points] Find explicit, real-valued solutions to each of the following, as indicated. For this problem, **USE Laplace transforms**.

a. [8 points] Find the solution $y(t)$ to the initial value problem $y'' + 3y' + 2y = 4$, $y(0) = 0$, $y'(0) = 0$.

Solution: Taking the Laplace transform of both sides of the equation, we have $\mathcal{L}\{y'' + 3y' + 2y\} = \frac{4}{s}$, so that, with $Y = \mathcal{L}\{y\}$,

$$s^2Y + 3sY + 2Y = \frac{4}{s},$$

so that

$$Y = \frac{4}{s(s^2 + 3s + 2)} = \frac{4}{s(s + 1)(s + 2)}.$$

Partial fractions allows us to rewrite $\frac{4}{s(s+1)(s+2)} = A\frac{1}{s} + B\frac{1}{s+1} + C\frac{1}{s+2}$, so that we can find the inverse transform

$$y(t) = \mathcal{L}^{-1}\{A\frac{1}{s} + B\frac{1}{s + 1} + C\frac{1}{s + 2}\} = A + Be^{-t} + Ce^{-2t}.$$

To find the constants A, B, and C, we solve in the equality giving the partial fractions decomposition. Clearing the denominators, we have $4 = A(s + 1)(s + 2) + Bs(s + 2) + Cs(s + 1)$. Plugging in $s = 0$, $A = 2$; with $s = -1$, $B = -4$; and with $s = -2$, $C = 2$. Thus

$$y = 2 - 4e^{-t} + 2e^{-2t}.$$

b. [7 points] Find the solution $z(t)$ to the initial value problem $z'' + 2z' + 10z = 0$, $z(0) = 1$, $z'(0) = 3$.

Solution: Proceding as above, the forward transform gives

$$(s^2Z - s - 3) + 2(sZ - 1) + 10Z = 0,$$

so that

$$Z = \frac{s + 5}{s^2 + 2s + 10} = \frac{s + 5}{(s + 1)^2 + 9}.$$

To find the inverse transform, we rewrite the right hand side as $\frac{s+5}{(s+1)^2+9} = \frac{s+1}{(s+1)^2+9} + \frac{4}{(s+1)^2+9}$. We can then invert both terms to get

$$z = \mathcal{L}^{-1}\{\frac{s + 1}{(s + 1)^2 + 9}\} + \mathcal{L}^{-1}\{\frac{4}{(s + 1)^2 + 9}\} = e^{-t}\cos(3t) + \frac{4}{3}e^{-t}\sin(3t).$$