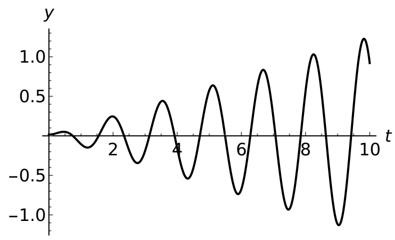
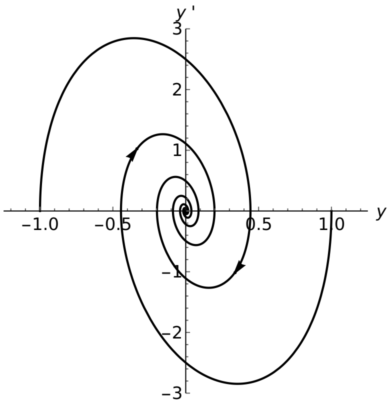
3. [14 points] In this problem we consider the differential equation $y'' + ky' + 16y = F_0 \cos(\omega t)$.

a. [7 points] If the solution to the problem is shown in the figure to the right when $F_0 = 1$, what can you say about the values of k and ω? Solve your equation and explain how your solution would give this graph.



Solution: The simplest guess is that we are seeing forcing of an undamped problem, so that $k = 0$, at the natural frequency of the system, which is $\omega = \sqrt{16} = 4$. In this case we're solving $y'' + 16y = \cos(4t)$. The complementary homogeneous solution is $y_c = c_1 \cos(4t) + c_2 \sin(4t)$. The particular solution will be $y_p = at\cos(4t) + bt\sin(4t)$, and because there are no odd derivatives in the problem we will find $a = 0$. Then $y_p' = b\sin(4t) + 4bt\cos(4t)$ and $y_p'' = 8b\cos(4t) - 16bt\sin(4t)$; plugging in, we get $8b\cos(4t) - 16bt\sin(4t) + 16bt\sin(4t) = \cos(4t)$, so that $b = \frac{1}{8}$. Thus the general solution is $y = c_1\cos(4t) + c_2\sin(4t) + \frac{1}{8}\, t\,\sin(4t)$, which will have a linearly growing solution as shown in the figure.

b. [7 points] Now suppose that when $F_0 = 0$ the phase portrait for the equation is shown to the right. Which of $k = -4$, $k = 6$, or $k = 10$ could we have used in this case? Solve the problem with that value of k and explain how your solution would give this graph.



Solution: We see that the eigenvalues are complex with negative real part. Solving the characteristic equation, we have $\lambda^2 + k\lambda + 16 = 0$, so that $\lambda = -\frac{k}{2} \pm \frac{1}{2}\sqrt{k^2 - 64}$. For this to have complex roots, $-8 < k < 8$, and for the real part to be negative, $k > 0$. Thus we require that $0 < k < 8$, and $k = 6$ is the only option of those provided that works. In this case, the roots of the equation are $\lambda = -3 \pm \frac{1}{2} i\sqrt{28} = -3 \pm i\sqrt{7}$, so that the general solution is $y = c_1 e^{-3t}\cos(\sqrt{7}\,t) + c_2 e^{-3t}\sin(\sqrt{7}\,t)$, which is a decaying oscillatory solution that will have inward spiral trajectories in the phase plane.

Alternately, we could see what the eigenvalues are for each of the indicated values of k: if $k = -4$, $\lambda^2 - 4\lambda + 16 = 0$, and $\lambda = 2 \pm i\frac{1}{2}\sqrt{48} = 2 \pm i2\sqrt{3}$, which would growing spiral solutions. If $k = 6$, $\lambda = -3 \pm i\sqrt{7}$, as shown above. If $k = 10$, $\lambda = -5 \pm 6 = 1, -11$. This will not have oscillatory solutions, and so cannot give the spiral shown.