4. [15 points] Note in this problem that $\mathcal{L}$ indicates the Laplace transform.

a. [5 points] If $f(t) = t\,e^{-t}$, use the integral definition of the Laplace transform to find $F(s) = \mathcal{L}\{f(t)\}$.

Solution: We have

$$\begin{aligned}\mathcal{L}\{f(t)\} = \int_0^\infty te^{-t}e^{-st}\,dt = \int_0^\infty te^{-(s+1)t}\,dt \\ = -\frac{1}{s+1}te^{-(s+1)t}\Big|_{t=0}^{t\to\infty} + \frac{1}{s+1}\int_0^\infty e^{-(s+1)t}\,dt \\ = 0 - \frac{1}{(s+1)^2}e^{-(s+1)t}\Big|_{t=0}^{t\to\infty} = \frac{1}{(s+1)^2}.\end{aligned}$$

b. [5 points] Use rules from the table of transforms to confirm your result in (a). Be sure that it is clear what rules you are using and how they give the result you obtain.

Solution: There are several ways we could approach this. First, we could note that with $g(t) = t$, $\mathcal{L}\{g(t)\} = \frac{1}{s^2} = G(s)$, so that $\mathcal{L}\{e^{-t}\cdot t\} = G(s+1) = \frac{1}{(s+1)^2} = \mathcal{L}\{te^{-t}\}$.

Alternately, we could note that with $g(t) = e^{-t}$, $\mathcal{L}\{g(t)\} = \frac{1}{s+1} = G(s)$. Then $\mathcal{L}\{tg(t)\} = -G'(s) = \frac{1}{(s+1)^2} = \mathcal{L}\{te^{-t}\}$.

c. [5 points] The solution to $y'' + 3y' + 2y = e^{-t}$ with initial conditions $y(0) = 0$, $y'(0) = 2$ is $y = e^{-t} - e^{-2t} + te^{-t}$. Transform the solution to the equation and the equation itself, and show that the two expressions you get for $Y(s) = \mathcal{L}\{y(t)\}$ are the same.

Solution: The transform of the differential equation gives $(-2+s^2Y)+3sY+2Y = \frac{1}{s+1}$, so that

$$Y = \left(\frac{2}{s^2+3s+2}\right) + \left(\frac{1}{(s+1)(s^2+3s+2}\right) = \frac{2(s+1)+1}{(s+1)^2(s+2)} = \frac{2s+3}{(s+1)^2(s+2)}.$$

The transform of the given solution is

$$\begin{aligned}\mathcal{L}\{y\} &= \frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{(s+1)^2} \\ &= \frac{(s+1)(s+2)-(s+1)^2+(s+2)}{(s+1)^2(s+2)} = \frac{2s+3}{(s+1)^2(s+2)},\end{aligned}$$

which is the same.

