5. [14 points] In lab 3 we considered the nonlinear system

$$N' = \gamma(A - N(1+P)), \qquad P' = P(N-1).$$

We established that the equilibrium solutions to the system are $(N, P) = (A, 0)$ and $(N, P) = (1, A-1)$, and that near the latter the system is approximated by the linear second order problem $v'' + \gamma v' + \gamma A(A-1)v = 0$, where v is the small variation in P from the equilibrium $A - 1$.

a. [4 points] Write the linear, second-order problem from above as a system of two linear, first-order equations.

Solution: Let $x = v$ and $y = v'$. Then $x' = y$ and $y' = -\gamma y - \gamma A(A-1)x$, or, if we prefer a matrix formulation, $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\gamma A(A-1) & -\gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

b. [6 points] Suppose that we pick A and γ so that the characteristic equation of the linear second-order equation has a repeated root. Find the solution to the linear second-order equation in this case, and use your solution to write the solution to the system you found in (a). *(If you are stuck, assume that $A = 2$ and find a nonzero γ to finish the problem with a one point penalty.)*

Solution: Note that the characteristic equation of the linear equation is $\lambda^2 + \gamma\lambda + \gamma A(A-1) = 0$, so that $\lambda = -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\gamma^2 - 4\gamma A(A-1)}$. If there is a repeated root, the argument of the square root vanishes, so that we are left with $\lambda = -\frac{\gamma}{2}$, repeated. The general solution is then $v = c_1 e^{-\gamma t/2} + c_2 t e^{-\gamma t/2}$. This is just the top, x, component of the solution to the system in (a). The bottom component is its derivative, $y = v' = -\frac{\gamma}{2} c_1 e^{-\gamma t/2} + c_2(1 - \frac{\gamma}{2} t)e^{-\gamma t/2}$. Thus the solution to the system is just $x = c_1 e^{-\gamma t/2} + c_2 t e^{-\gamma t/2}$, $y = -\frac{\gamma}{2} c_1 e^{-\gamma t/2} + c_2(1 - \frac{\gamma}{2} t)e^{-\gamma t/2}$, or, in matrix form,

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -\frac{\gamma}{2} \end{pmatrix} e^{-\gamma t/2} + c_2 \left(\begin{pmatrix} 1 \\ -\frac{\gamma}{2} \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) e^{-\gamma t/2}.$$

With $A = 2$, we have $\lambda^2 + \gamma\lambda + 2\gamma = 0$, so that $\lambda = -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\gamma^2 - 8\gamma} = -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\gamma(\gamma - 8)}$, so we have a repeated root if $\gamma = 8$ (or $\gamma = 0$, but we do not consider that). In this case $\lambda = -4$, so that $y = c_1 e^{-4t} + c_2 t e^{-4t}$.

c. [4 points] In Part B of the lab, we assumed that A was a function of time, that is, $A = A(t) = A_0 + 2a\cos(\omega t)$. Suppose instead we picked $A(t) = A_0 \tan(\omega t)$, so that $v'' + \gamma v' + q(t)v = 0$, with $q(t) = \gamma A(t)(A(t) - 1)$. If we start with $v(0) = 0.5$, $v'(0) = 0$, what is the longest interval on which the solution to the initial value problem is certain to have a unique solution, and why? (Note that you cannot solve the equation by hand.)

Solution: We know that there will be a unique solution everywhere the coefficients of the equation are continuous. In this case the only problem is where $q(t) = \gamma A_0 \tan(\omega t)(A_0 \tan(\omega t) - 1)$ is discontinuous, which is where $t = \frac{n\pi}{2\omega}$ (for any odd integer n). Thus we are certain of a unique solution for $0 \le t < \frac{\pi}{2\omega}$.

