6. [15 points] Consider a physical system modeled by the differential equation

\[ x'' + \gamma x' + kx = f(t), \]

where \( x(t) \) is the physical quantity being measured and \( \gamma \) and \( k \) are constants.

a. [4 points] If the physical system is underdamped, what can you say about the parameters \( \gamma \) and \( k \)?

**Solution:** If the system is underdamped, we know that the roots of the characteristic equation are complex. The characteristic equation is \( \lambda^2 + \gamma \lambda + k = 0 \), so that \( \lambda = \frac{1}{2}(-\gamma \pm \sqrt{\gamma^2 - 4k}) \), so we know that \( \gamma^2 < 4k \). Because this is a physical system we know that both \( \gamma \) and \( k \) should be positive; clearly this condition also requires that \( k > 0 \).

b. [5 points] If \( x(0) = x_0, \ x'(0) = v_0 \), and \( \mathcal{L}\{f(t)\} = F(s) \), find the transform \( X(s) = \mathcal{L}\{x(t)\} \).

**Solution:** Applying the Laplace transform to both sides of the differential equation, we have

\[ s^2X - x_0s - v_0 + \gamma sX - \gamma x_0 + kX = F(s), \]

so that

\[ X = \frac{x_0s + v_0 + \gamma x_0}{s^2 + \gamma s + k} + \frac{F(s)}{s^2 + \gamma s + k}. \]

c. [6 points] If \( f(t) = 0 \), assuming as in (a) that the system is underdamped, invert your transform from (b) to find \( x(t) \). (If you are stuck, assume the equation is \( x'' + \gamma x' + \gamma^2 x = 0 \).)

**Solution:** If the system is underdamped, then we know that the characteristic polynomial \( \lambda^2 + \gamma \lambda + k = (\lambda + \frac{\gamma}{2})^2 + k - \frac{\gamma^2}{4} \). Then, with \( b = \sqrt{k - \frac{\gamma^2}{4}} \) and \( f(t) = 0 \), we have

\[ x = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{ \frac{x_0(s + \frac{\gamma}{2})}{(s + \frac{\gamma}{2})^2 + b^2} + \frac{v_0 + \gamma x_0}{(s + \frac{\gamma}{2})^2 + b^2} \right\} \]

\[ = x_0e^{-\gamma t/2} \cos(bt) + \frac{1}{b}(v_0 + \gamma x_0)e^{-\gamma t/2} \sin(bt). \]

For the hint, the characteristic polynomial is \( p(s) = s^2 + \gamma s + \gamma^2 = (s + \frac{\gamma}{2})^2 + \frac{3\gamma^2}{4} \), so that we have the answer above with \( b = \frac{\sqrt{3} \gamma}{2} \).