6. [15 points] Consider a physical system modeled by the differential equation

$$x'' + \gamma x' + kx = f(t),$$

where $x(t)$ is the physical quantity being measured and γ and k are constants.

a. [4 points] If the physical system is underdamped, what can you say about the parameters γ and k?

Solution: If the system is underdamped, we know that the roots of the characteristic equation are complex. The characteristic equation is $\lambda^2 + \gamma\lambda + k = 0$, so that $\lambda = \frac{1}{2}(-\gamma \pm \sqrt{\gamma^2 - 4k})$, so we know that $\gamma^2 < 4k$. Because this is a physical system we know that both γ and k should be positive; clearly this condition also requires that $k > 0$.

b. [5 points] If $x(0) = x_0$, $x'(0) = v_0$, and $\mathcal{L}\{f(t)\} = F(s)$, find the transform $X(s) = \mathcal{L}\{x(t)\}$.

Solution: Applying the Laplace transform to both sides of the differential equation, we have

$$s^2X - x_0 s - v_0 + \gamma sX - \gamma x_0 + kX = F(s),$$

so that

$$X = \frac{x_0 s + v_0 + \gamma x_0}{s^2 + \gamma s + k} + \frac{F(s)}{s^2 + \gamma s + k}.$$

c. [6 points] If $f(t) = 0$, assuming as in (a) that the system is underdamped, invert your transform from (b) to find $x(t)$. *(If you are stuck, assume the equation is $x'' + \gamma x' + \gamma^2 x = 0$.)*

Solution: If the system is underdamped, then we know that the characteristic polynomial $\lambda^2 + \gamma\lambda + k = (\lambda + \frac{\gamma}{2})^2 + k - \frac{\gamma^2}{4}$. Then, with $b = \sqrt{k - \frac{\gamma^2}{4}}$ and $f(t) = 0$, we have

$$\begin{aligned} x = \mathcal{L}^{-1}\{X(s)\} &= \mathcal{L}^{-1}\{\frac{x_0 s + v_0 + \gamma x_0}{(s + \frac{\gamma}{2})^2 + b^2}\} \\ &= \mathcal{L}^{-1}\{\frac{x_0(s + \frac{\gamma}{2})}{(s + \frac{\gamma}{2})^2 + b^2} + \frac{v_0 + \frac{\gamma}{2}x_0}{(s + \frac{\gamma}{2})^2 + b^2}\} \\ &= x_0 e^{-\gamma t/2}\cos(bt) + \frac{1}{b}(v_0 + \frac{\gamma}{2}x_0)e^{-\gamma t/2}\sin(bt). \end{aligned}$$

For the hint, the characteristic polynomial is $p(s) = s^2 + \gamma s + \gamma^2 = (s + \frac{\gamma}{2})^2 + \frac{3\gamma^2}{4}$, so that we have the answer above with $b = \frac{\sqrt{3}\gamma}{2}$.