3. In a lab there are five identical "RLC" circuits in each of which the capacitor charge satisfies $LQ'' + RQ' + C^{-1}Q = E(t)$ with $L = 3$ Henries, $R = 4$ Ohms, $C = 0.5$ Farads. The five circuits are driven by five time-dependent voltage sources $E(t)$ and have different amounts $Q(0)$ of initial charge on the capacitor and different amounts $Q'(0)$ of current flowing at time zero, according to this table:

Circuit	Capacitor charge	Voltage source $E(t)$	$Q(0)$	$Q'(0)$
1	$Q = Q_1(t)$	$E_1(t) = 1/(1+t^2)$ Volts	0.25 Coul.	0 Amp.
2	$Q = Q_2(t)$	$E_2(t) = 1/(1+t^2)$ Volts	0 Coul.	1 Amp.
3	$Q = Q_3(t)$	$E_3(t) \equiv 6$ Volts	0.25 Coul.	0 Amp.
4	$Q = Q_4(t)$	$E_4(t) = (25+24t^2)/(1+t^2)$ Volts	1 Coul.	1 Amp.
5	$Q = Q_5(t)$	For $E_5(t)$, see part (d)	0 Coul.	0 Amp.

(a) (3 Points.) Express $Q_4(t)$ in terms of $Q_1(t)$, $Q_2(t)$, and/or $Q_3(t)$.

Solution: we apply the superposition principle for nonhomogeneous linear ODE, noticing that each equation has exactly the same linear operator $LD^2 + RD + C^{-1}$ on the left-hand side. So, we write $Q_4(t) = aQ_1(t) + bQ_2(t) + cQ_3(t)$ and try to determine a, b, c. The initial condition $Q_4(0) = 1$ then reads $\frac{1}{4}a + \frac{1}{4}c = 1$, and $Q_4'(0) = 1$ reads $b = 1$. To fully determine a, c, we require that the differential equation for $Q_4(t)$ holds, which means that

$$\frac{25+24t^2}{1+t^2} = E_4(t) = aE_1(t) + bE_2(t) + cE_3(t) = \frac{a}{1+t^2} + \frac{1}{1+t^2} + 6c.$$

The numerator over the common denominator of $1 + t^2$ gives us the equation $24(1 + t^2) = a + 6c(1 + t^2)$, so from the coefficients of t^2 we get that $c = 4$. Taking the constant terms, or going back to $Q_4(0) = 1$, one finds that $a = 0$. So $a = 0$, $b = 1$, and $c = 4$, and therefore by superposition,

$$\boxed{Q_4(t) = Q_2(t) + 4Q_3(t).}$$

(b) (2 Points.) Find $\lim_{t\to+\infty}[Q_1(t) - Q_2(t)]$.

Solution: The difference $\Delta Q(t) := Q_1(t) - Q_2(t)$ is a solution of the associated homogeneous equation $L\Delta Q'' + R\Delta Q' + C^{-1}\Delta Q = 0$. Since the coefficients are all positive, $\Delta Q(t)$ is a transient response that decays to zero as $t \to +\infty$. It is not necessary to know what the initial conditions are, nor is it necessary to use the fact that the forcing function is $E(t) = 1/(1 + t^2)$. We conclude that

$$\boxed{\lim_{t\to+\infty}[Q_1(t) - Q_2(t)] = 0.}$$

(c) (2 Points.) Would the answer to part (b) be different if $R = 0$ Ohms instead? Why or why not?

Solution: yes it would be different, because then the difference $Q_1(t) - Q_2(t)$ satisfies $L\Delta Q'' + C^{-1}\Delta Q = 0$, the general solution of which is simple harmonic motion with natural frequency of vibration $\Omega_0 = 1/\sqrt{LC}$; this does not decay to zero as $t \to +\infty$, so in general the limit of $\Delta Q(t)$ as $t \to +\infty$ does not exist.

(d) (3 Points.) Circuit number 5 has a "pulsed" voltage source $E(t)$ that is zero except on the time interval $1 < t < 6$ seconds, at the beginning of which it is suddenly switched on to 10 Volts, and during which it increases exponentially following $E(t) = 10e^{b(t-1)}$ for some rate b sec^{-1}. If instantaneously after switching on, $E'(1) = 30$ Volts/sec, use the definition of the Laplace transform to find $\mathscr{L}\{E(t)\}$ (but don't solve for $Q_5(t)$).

Solution: over the time interval $1 < t < 6$, we have $E(t) = 10e^{b(t-1)}$, and since this implies that $E'(1) = 10b$, we need $b = 3$ sec^{-1} to match a rate of change of 30 Volts/sec. Then, using the definition,

$$\mathscr{L}\{E(t)\} = \int_1^6 E(t)\mathrm{e}^{-st}\,\mathrm{d}t = \int_1^6 10\mathrm{e}^{3(t-1)}\mathrm{e}^{-st}\,\mathrm{d}t = 10\mathrm{e}^{-3}\int_1^6 \mathrm{e}^{(3-s)t}\,\mathrm{d}t = 10\mathrm{e}^{-3}\frac{\mathrm{e}^{(3-s)6} - \mathrm{e}^{(3-s)1}}{3-s}.$$