4. (5 Points.) Using the method of undetermined coefficients, solve the initial-value problem $y'' - 4y = 12e^{-2t}$, $y(0) = 0$, $y'(0) = 1$ for $y(t)$.

Solution: first we solve the associated homogeneous problem: $y'' - 4y = 0$. The characteristic equation reads $\lambda^2 - 4 = 0$ so $\lambda = 2, -2$. The complementary function is therefore $y_c(t) = c_1e^{2t} + c_2e^{-2t}$. Next we find a particular solution using the method of undetermined coefficients. Because e^{-2t} is a solution of the associated homogeneous problem, we assume the form $y = Y(t) = Ate^{-2t}$. Then, some computation gives:

$$Y'(t) = Ae^{-2t} - 2Ate^{-2t},$$

$$Y''(t) = -4Ae^{-2t} + 4Ate^{-2t},$$

so substitution gives $[-4Ae^{-2t} + 4Ate^{-2t}] - 4[Ate^{-2t}] = 12e^{-2t}$ so $A = -3$. The general solution is therefore

$$y = y_c(t) + Y(t) = -3te^{-2t} + c_1e^{2t} + c_2e^{-2t}.$$

Likewise,

$$y' = -3e^{-2t} + 6te^{-2t} + 2c_1e^{2t} - 2c_2e^{-2t}.$$

Therefore, $y(0) = c_1 + c_2$ and $y'(0) = -3 + 2c_1 - 2c_2$. Imposing the initial conditions then gives $c_1 = 1$ and $c_2 = -1$. The solution is

$$\boxed{y(t) = -3te^{-2t} + e^{2t} - e^{-2t}.}$$