5. (6 Points.) The differential equation $t^2y'' - 2ty' + 2y = 0$ has the following two solutions: $y = t$ and $y = t^2$. Assuming that $t > 0$, solve the initial-value problem $t^2y'' - 2ty' + 2y = t^2$, $y(1) = 1$, $y'(1) = 0$.

Solution: we know that the general solution is $y(t) = y_c(t) + Y(t)$ where $y_c(t) = c_1t + c_2t^2$ is the complementary function (because t and t^2 are linearly independent, in fact $W[t, t^2](t) = t^2$ which is nonzero on the interval $(0, +\infty)$ containing the initial point $t = 1$). To find a particular solution, we need to use the method of variation of parameters. Putting the problem in standard form (divide by t^2), it reads $y'' - 2t^{-1}y' + 2t^{-2}y = g(t) = 1$. Looking for a particular solution in the form $Y(t) = u_1(t)t + u_2(t)t^2$, we solve the system

$$\begin{aligned} tu_1'(t) + t^2u_2'(t) &= 0 \\ u_1'(t) + 2tu_2'(t) &= g(t) = 1 \end{aligned}$$

for $u_1'(t)$ and $u_2'(t)$. We get $u_1'(t) = -1$ and $u_2'(t) = t^{-1}$. Any antiderivatives will do to obtain a particular solution, so we take $u_1(t) = -t$ and $u_2(t) = \ln(|t|)$. So the particular solution we get is $Y(t) = -t^2 + t^2\ln(|t|)$. The general solution is therefore

$$y(t) = -t^2 + t^2\ln(|t|) + c_1t + c_2t^2,$$

and taking a derivative,

$$y'(t) = -2t + t + 2t\ln(|t|) + c_1 + 2c_2t.$$

Therefore, $y(1) = -1 + c_1 + c_2$ and $y'(1) = -1 + c_1 + 2c_2$. Imposing the initial conditions then gives $c_1 = 3$ and $c_2 = -1$, so

$$\boxed{\begin{aligned} y(t) &= -t^2 + t^2\ln(|t|) + 3t - t^2 \\ &= -2t^2 + t^2\ln(|t|) + 3t. \end{aligned}}$$

In all of these formulas, we could have written $\ln(t)$ instead of $\ln(|t|)$ because $t > 0$.