6. (4 Points.) Consider the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} g(t) \\ 0 \end{pmatrix}, \quad \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

and assume that $x(t)$ and $y(t)$ satisfy the initial conditions $x(0) = 0$ and $y(0) = 1$. Let $g(t)$ be a function having a Laplace transform denoted $G(s)$, for s large enough. Find $X(s) = \mathscr{L}\{x(t)\}$ and $Y(s) = \mathscr{L}\{y(t)\}$ in terms of $G(s)$. Your answers should be in terms of s.

The equations of the system read:

$$\begin{aligned} x' &= x + 2y + g(t) \\ y' &= 2x + y \end{aligned}$$

Taking the Laplace transform of both equations using the initial conditions:

$$\begin{aligned} sX(s) &= X(s) + 2Y(s) + G(s) \\ sY(s) - 1 &= 2X(s) + Y(s) \end{aligned}$$

We rearrange this as a linear system for $X(s)$ and $Y(s)$:

$$\begin{pmatrix} 1-s & 2 \\ 2 & 1-s \end{pmatrix} \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} -G(s) \\ -1 \end{pmatrix}.$$

Taking the inverse matrix (using the fact that its determinant is $s^2 - 2s - 3 = (s-3)(s+1)$) we have

$$\begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = \frac{1}{(s-3)(s+1)} \begin{pmatrix} 1-s & -2 \\ -2 & 1-s \end{pmatrix} \begin{pmatrix} -G(s) \\ -1 \end{pmatrix}.$$

So,

$$\boxed{X(s) = \frac{(s-1)G(s)+2}{(s-3)(s+1)}, \quad Y(s) = \frac{2G(s)+s-1}{(s-3)(s+1)}.}$$