1. [15 points] For this problem note that the general solution to $y'' + 5y' + 4y = 0$ is $y = c_1 e^{-t} + c_2 e^{-4t}$. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] Find a real-valued general solution to

$$y'' + 5y' + 4y = 3e^{-4t}.$$

Solution: We know the general solution is $y = y_c + y_p$. We use the Method of Undetermined Coefficients to find y_p, guessing $y_p = Ate^{-4t}$, after multiplying our first guess ($y_p = Ae^{-4t}$) by t because the forcing term is present in our homogeneous solution. Then $y'_{p2} = Ae^{-4t} - 4Ate^{-4t}$ and $y''_{p2} = -8Ae^{-4t} + 4Ate^{-4t}$, so that on plugging in we get

$$(-8A + 5A)e^{-4t} = 3e^{-4t},$$

so that $-3A = 3$, and $A = -1$.
Thus the general solution is

$$y = c_1 e^{-t} + c_2 e^{-4t} - te^{-4t}.$$

If we use Variation of Parameters, we have $u'_1 e^{-t} + u'_2 e^{-4t} = 0$ and $-u'_1 e^{-t} - 4u'_2 e^{-4t} = 3e^{-4t}$. Solving, we find $u'_2 = -1$ and $u'_1 = e^{-3t}$, so that $u_1 = -\frac{1}{3}e^{-3t}$ and $u_2 = -t$, and $y_p = -\frac{1}{3}e^{-4t} - te^{-4t}$.

b. [8 points] Find the solution to the

$$y'' + 5y' + 4y = 16t, \qquad y(0) = 2, \quad y'(0) = -2.$$

Solution: We know the general solution is $y = y_c + y_p$. We use the Method of Undetermined Coefficients to find y_p, guessing $y_p = A + Bt$. Plugging in,

$$5B + 4A + 4Bt = 16t,$$

so that $B = 4$ and $A = -5$. Thus the general solution is $y = c_1 e^{-t} + c_2 e^{-4t} - 5 + 4t$.
Applying the initial conditions, we have

$$y(0) = c_1 + c_2 - 5 = 2, \quad \text{and}$$
$$y'(0) = -c_1 - 4c_2 + 4 = -2.$$

Thus $c_1 + c_2 = 7$ and $-c_1 - 4c_2 = -6$. Adding the two, we have $-3c_2 = 1$, so $c_2 = -1/3$. Then the first gives $c_1 = 22/3$, and our solution is

$$y = \frac{22}{3}e^{-t} - \frac{1}{3}e^{-4t} - 5 + 4t.$$

We can, of course find y_p with Variation of Parameters. Then $u'_1 e^{-t} + u'_2 e^{-4t} = 0$ and $-u'_1 e^{-t} - 4u'_2 e^{-4t} = 16t$. Solving, we find $u'_2 = -\frac{16}{3}te^{4t}$, so that $u_2 = (\frac{1}{3} - \frac{4}{3}t)e^{4t}$ and $u'_1 = \frac{16}{3}te^{t}$, so that $u_1 = \frac{16}{3}(-1 + t)e^{t}$. Then $y_p = -5 + 4t$, as before.