1. [14 points] Find real-valued solutions for each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] Solve $\frac{1}{3}y'' + 2y' + 3y = 2t$, $y(0) = 0$, $y'(0) = \frac{4}{3}$.

Solution: The algebra may be easier if we first multiply by 3, obtaining $y''+6y'+9y = 6t$. The characteristic equation for this is $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0$, so $\lambda = -3$ twice, and the homogeneous solution is $y_c = c_1e^{-3t} + c_2te^{-3t}$. To find y_p we use the method of undetermined coefficients, guessing $y_p = At + B$. Then, plugging in,

$$6A + 9At + 9B = 6t,$$

so that $A = \frac{2}{3}$ and $B = -\frac{4}{9}$. The general solution is

$$y = c_1e^{-3t} + c_2te^{-3t} + \frac{2}{3}\,t - \frac{4}{9}.$$

Applying the initial conditions, we have $y(0) = c_1 - \frac{4}{9} = 0$, so $c_1 = \frac{4}{9}$, and $y'(0) = -3c_1 + c_2 + \frac{2}{3} = c_2 - \frac{2}{3} = \frac{4}{3}$, so that $c_2 = 2$. Thus

$$y = \frac{4}{9}\,e^{-3t} + 2te^{-3t} + \frac{2}{3}\,t - \frac{4}{9}.$$

b. [7 points] Find the general solution to $y'' + 2y' + 5y = 2te^{-t}$.

Solution: The general solution will be $y = y_c + y_p$, where y_c solves the complementary homogenous problem and y_p is a particular solutions. For y_c we guess $y = e^{\lambda t}$, so that $\lambda^2 + 2\lambda + 5 = (\lambda + 1)^2 + 4 = 0$, and $\lambda = -1 \pm 2i$. Thus $y_c = c_1e^{-t}\cos(2t) + c_2e^{-t}\sin(2t)$. For y_p we use the method of undetermined coefficients, taking $y_p = (At+B)e^{-t}$. Plugging in, we have

$$(At - 2A + B)e^{-t} + 2(-At + A - B)e^{-t} + 5(At + B)e^{-t} = 3te^{-t}.$$

Collecting terms in e^{-t} and te^{-t}, we have $4B = 0$ and $4A = 2$. Thus $B = 0$ and $A = \frac{1}{2}$, and

$$y = c_1e^{-t}\cos(2t) + c_2e^{-t}\sin(2t) + \frac{1}{2}\,te^{-t}.$$