1. [14 points] Find real-valued solutions for each of the following, as indicated. (Note that minimal partial credit will be given on this problem.)

   a. [7 points] Solve \( \frac{1}{3} y'' + 2y' + 3y = 2t, \ y(0) = 0, \ y'(0) = \frac{4}{3}. \)

   Solution: The algebra may be easier if we first multiply by 3, obtaining \( y'' + 6y' + 9y = 6t. \) The characteristic equation for this is \( \lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0, \) so \( \lambda = -3 \) twice, and the homogeneous solution is \( y_c = c_1 e^{-3t} + c_2 te^{-3t}. \) To find \( y_p \) we use the method of undetermined coefficients, guessing \( y_p = At + B. \) Then, plugging in,

   \[
   6A + 9At + 9B = 6t,
   \]

so that \( A = \frac{2}{3} \) and \( B = -\frac{4}{9}. \) The general solution is

   \[
   y = c_1 e^{-3t} + c_2 te^{-3t} + \frac{2}{3} t - \frac{4}{9}.
   \]

Applying the initial conditions, we have \( y(0) = c_1 - \frac{4}{9} = 0, \) so \( c_1 = \frac{4}{9}, \) and \( y'(0) = -3c_1 + c_2 + \frac{2}{3} = c_2 - \frac{2}{3} = \frac{1}{3}, \) so that \( c_2 = 2. \) Thus

   \[
   y = \frac{4}{9} e^{-3t} + 2te^{-3t} + \frac{2}{3} t - \frac{4}{9}.
   \]

b. [7 points] Find the general solution to \( y'' + 2y' + 5y = 2te^{-t}. \)

   Solution: The general solution will be \( y = y_c + y_p, \) where \( y_c \) solves the complementary homogenous problem and \( y_p \) is a particular solutions. For \( y_c \) we guess \( y = e^{\lambda t}, \) so that \( \lambda^2 + 2\lambda + 5 = (\lambda + 1)^2 + 4 = 0, \) and \( \lambda = -1 \pm 2i. \) Thus \( y_c = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t). \) For \( y_p \) we use the method of undetermined coefficients, taking \( y_p = (At + B)e^{-t}. \) Plugging in, we have

   \[
   (At - 2A + B)e^{-t} + 2(-At + A - B)e^{-t} + 5(At + B)e^{-t} = 3te^{-t}.
   \]

Collecting terms in \( e^{-t} \) and \( te^{-t}, \) we have \( 4B = 0 \) and \( 4A = 2. \) Thus \( B = 0 \) and \( A = \frac{1}{2}, \) and

   \[
   y = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t) + \frac{1}{2} te^{-t}.
   \]