2. [14 points] Find each of the following, providing an explicit formula where appropriate. *(Note that minimal partial credit will be given on this problem.)*

a. [5 points] $Y(s) = \mathcal{L}\{y(t)\}$ if $y'' + 4y' + 20y = 3\sin(2t)$, $y(0) = 1$, $y'(0) = 2$.

Solution: Transforming both sides of the equation, we have

$$s^2Y - s - 2 + 4(sY - 1) + 20Y = \frac{6}{s^2+4},$$

so that

$$Y = \frac{s+6}{s^2+4s+20} + \frac{6}{(s^2+4)(s^2+4s+20)}.$$

b. [5 points] $\mathcal{L}^{-1}\{\frac{s}{s^2+4s+5}\}$

Solution: This is

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{s}{s^2+4s+5}\} &= \mathcal{L}^{-1}\{\frac{(s+2)-2}{(s+2)^2+1}\} \\ &= \mathcal{L}^{-1}\{\frac{s+2}{(s+2)^2+1}\} - \mathcal{L}^{-1}\{\frac{2}{(s+2)^2+1}\} \\ &= e^{-2t}\cos(t) - 2e^{-2t}\sin(t).\end{aligned}$$

c. [4 points] Using the integral definition of the Laplace transform, derive the transform rule $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-sc}F(s)$ for a function $f(t)$ with transform $L\{f(t)\} = F(s)$. (Recall $u_c(t)$ is the unit step function at $t = c$, $u_c(t) = \begin{cases} 0, & 0 < t < c \\ 1, & t \geq c \end{cases}$.)

Solution: The integral definition is $\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt$. Noting that $u_c(t)$ is zero for $t < c$, we may rewrite this as an integral with lower bound $t = c$. With the substitution $w = t - c$, we have

$$\begin{aligned}\int_0^\infty e^{-st}u_c(t)f(t-c)\,dt = \int_c^\infty e^{-st}f(t-c)\,dt &= \int_0^\infty e^{-s(w+c)}f(w)\,dw \\ &= e^{-cs}\int_0^\infty e^{-sw}f(w)\,dw = e^{-cs}F(s).\end{aligned}$$