3. [14 points] Use Laplace transforms to solve each of the following.

a. [7 points] $y'' + 4y' + 4y = 2e^{-2t}$, $y(0) = 1$, $y'(0) = 0$.

Solution: Taking the Laplace transform of both sides of the equation, we have with $Y = \mathcal{L}\{y\}$,

$$s^2Y - s + 4sY - 4 + 4Y = \frac{2}{s+2}, \quad \text{or} \quad Y = \frac{2}{(s+2)^3} + \frac{s+4}{(s+2)^2}.$$

From transforms 3 and C from the table, we see that the first term in Y will invert as $\mathcal{L}^{-1}\{\frac{2}{(s+2)^3}\} = t^2\, e^{-2t}$. To do the second, we use partial fractions: $\frac{s+4}{(s+2)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2}$. Clearing the denominators, we have $s + 4 = A(s + 2) + B$, so that with $s = -2$ we find $B = 2$. Then $s = 0$ requires that $A = 1$, so that $\mathcal{L}^{-1}\{\frac{s+4}{(s+2)^2}\} = \mathcal{L}^{-1}\{\frac{1}{s+2} + \frac{2}{(s+2)^2}\} = e^{-2t} + 2te^{-2t}$. Combining this with the first result, we have

$$y = \mathcal{L}^{-1}\{Y\} = t^2e^{-2t} + e^{-2t} + 2te^{-2t}.$$

b. [7 points] $y'' + 3y' = \begin{cases} 12, & 0 \le t < 2 \\ 0, & t \ge 2 \end{cases}$, $y(0) = 0$, $y'(0) = 0$.

Solution: We want to transform both sides of the equation; the right-hand side we can do by using the definition of the transform, or by noting that the differential equation may be written as $y'' + 3y' = 12 - 12u_2(t)$. Transforming the equation using transform 6 in the table, we have

$$s^2Y + 3sY = \frac{12}{s} - \frac{12e^{-2s}}{s}, \quad \text{so that} \quad Y = \frac{12}{s^2(s+3)} - \frac{12e^{-2s}}{s^2(s+3)}.$$

To find y we need to invert the transform of $\frac{12}{s^2(s+3)}$. We decompose this with partial fractions: $\frac{12}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$. Clearing the denominators, $12 = As(s + 3) + B(s + 3) + Cs^2$. If $s = 0$, $B = 4$; if $s = -3$, $C = \frac{4}{3}$. Then, if $s = -2$, $12 = -2A + 4 + \frac{16}{3}$, so that $A = -\frac{4}{3}$. Thus $\mathcal{L}^{-1}\{\frac{12}{s^2(s+3)}\} = \mathcal{L}^{-1}\{-\frac{4}{3s} + \frac{4}{s^2} + \frac{4}{3(s+3)}\} = -\frac{4}{3} + 4t + \frac{4}{3}e^{-3t}$, and, using this and transform 6 from the table, we have

$$y = \mathcal{L}^{-1}\{Y\} = -\frac{4}{3} + 4t + \frac{4}{3}e^{-3t} - \left(-\frac{4}{3} + 4(t-2) + \frac{4}{3}e^{-3(t-2)}\right) u_2(t).$$