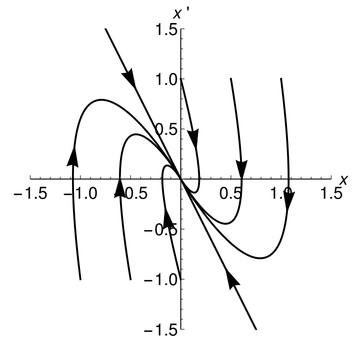
4. [14 points] Consider a mass-spring system modeled by

$$x'' + 4x' + \alpha x = 0.$$

a. [5 points] Suppose that the phase portrait for the system is that shown to the right, below. For what values of α, if any, will the system have this type of behavior? Explain.

Solution: The characteristic equation of the equation is $\lambda^2+4\lambda+\alpha = (\lambda+2)^2+\alpha-4 = 0$. The behavior shown in the phase portrait is that of a critically damped system, with a repeated eigenvalue and single eigenvector. This occurs when $\alpha = 4$. Thus we conclude that $\alpha = 4$.



b. [3 points] For what values of α, if any, will the system be underdamped? Critically damped? Overdamped? Explain how you obtain your answers.

Solution: The work from **(a)** shows that for $\alpha = 4$ the system is critically damped. It is underdamped when it has a complex conjugate pair of roots, which will occur when $\alpha > 4$. It is overdamped when $\alpha < 4$.

c. [6 points] Let $\alpha = 6$. How will the phase portrait for the system in this case differ from that given in **(a)**? Sketch the phase portrait for this case. In a separate graph, sketch representative solutions $x(t)$ as functions of time for the case $\alpha = 4$. (Note that you do not need to solve the problem to do this.)

Solution: From the work above, when $\alpha = 6$ the system will be underdamped. We can also see from the critically damped case in part (a) that the critical point $(0, 0)$ will be a sink and that trajectories will spiral in to the origin in a clockwise direction. This is shown in the figure to the right. When $\alpha = 4$ and we have the phase portrait shown in (a), the solution curves must converge to zero, with the possibility of crossing the t-axis (but not repeatedly), as shown in the lower figure.