5. [15 points] For each of the following, identify the statement as true or false by circling "True" or "False" as appropriate, and provide a short (one or two sentence) explanation indicating why that answer is correct.

a. [3 points] For the system $x' = -xy + y^2$, $y' = x^2 - xy$, the nonlinear trajectory in the phase plane with $x(0) = -3$ and $y(0) = 0$ lies on a circle centered on the origin.

[True] False

Solution: The system gives $\frac{dy}{dx} = \frac{y'}{x'} = \frac{x(x-y)}{-y(x-y)} = -\frac{x}{y}$. Separating and integrating, we get $y\,dy = -x\,dx$, so that $x^2 + y^2 = 2c$.

b. [3 points] For a linear differential operator $L = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)$, if y_1 and y_2 are different functions satisfying $L[y_1] = L[y_2] = g(t) \neq 0$, then, for any constants c_1 and c_2, $y = c_1y_1 - c_2y_2$ satisfies $L[y] = 0$.

True [False]

Solution: Relying on the linearity of the operator,

$$L[y] = L[c_1y_1 - c_2y_2] = c_1L[y_1] - c_2L[y_2] = c_1g(t) - c_2g(t) = (c_1 - c_2)g(t),$$

which is zero only if $c_1 - c_2 = 0$.

c. [3 points] The solution to a differential equation $my'' + ky = F(t)$ modeling the motion y of an undamped mechanical spring system with a periodic external force $F(t) = F_0\cos(\omega t)$ can always be written as $y = A\cos(\omega_0 t - \delta_1) + B\cos(\omega t - \delta_2)$, a sum of two oscillatory terms. (A, B, ω_0, δ_1 and δ_2 are constants.)

True [False]

Solution: This is only true if the forcing frequence ω is not equal to the natural frequency of the system, $\omega_0 = \sqrt{k/m}$. If $\omega = \omega_0$, we will have a growing solution $y = A\cos(\omega_0 t - \delta_1) + Bt\cos(\omega_0 - \delta_2)$.

d. [3 points] If $\lambda^2 + p\lambda + q = 0$ is the characteristic equation of a constant-coefficient linear differential equation $L[y] = g(t)$, then solving for $Y(s) = \mathcal{L}\{y(t)\}$ will result in an expression involving a product of $(s^2 + ps + q)^{-1}$ with other terms.

[True] False

Solution: The characteristic equation tells us that the differential equation is $L[y] = y'' + py' + qy = g(t)$. The transform of this is $s^2Y - sy(0) - y'(0) + psY - y(0) + qY = G(s)$, and solving for Y will give the indicated result.

e. [3 points] If $f(t) \neq 0$ has Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t), & 0 < t < c \\ 0, & t \geq c \end{cases}$, then $\mathcal{L}\{g(t)\} = (1 - e^{-sc})F(s)$.

True [False]

Solution: The easiest way to see this is to look for $h(t) = \mathcal{L}^{-1}\{(1 - e^{-sc})F(s)\} = f(t) - f(t-c)u_c(t)$. Then, by definition, $g(t) = (1 - u_c(t))f(t)$, and in general $h(t) \neq g(t)$.