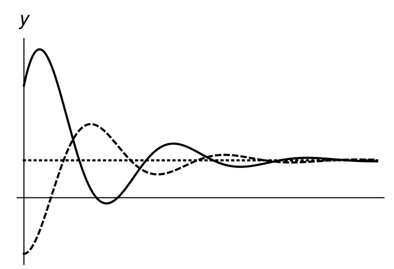
6. [14 points] In the following, we consider the behavior of solutions to a linear, second-order, constant-coefficient differential equation with a forcing term.

a. [5 points] Write a differential equation of this type that could have the three solution curves given to the right. Explain how you know your answer is correct.



Solution: The constant solution shows that we have a non-zero equilibrium solution, so the forcing term is $g(t) = k$, a constant. Then the two non-constant solutions show an oscillatory transient with decaying amplitude, so the characteristic equation of the differential equation must have complex roots with a negative real part. Thus any equation of the form $y'' + ay' + by = k$, where $a > 0$ and $a^2 - 4b < 0$ (and, because the equilibrium is positive, $k > 0$) will produce the desired result. One such is $y'' + y' + y = 1$.

b. [6 points] Now suppose that the general solution to the problem is $y = (c_1 + c_2 t + t\ln(t))e^{-t}$. What is the differential equation, including the forcing term? Explain.

Solution: Because the problem is linear we know that the general solution has the form $y = y_c + y_p$, where y_c is the solution to the complementary homogeneous problem and y_p is a solution to the problem with forcing. Because it is constant-coefficient and second-order, the solution y_c has terms of the form $e^{\lambda t}$ or $te^{\lambda t}$ (where λ may be zero or complex), so the homogeneous solution here must be $y_c = c_1 e^{-t} + c_2 te^{-t}$. This requires that $\lambda = -1$, twice, so the characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$, and the linear differential operator is $L[y] = y'' + 2y' + y$. Then $y_p = t\ln(t)e^{-t}$. We can find $g(t)$ by plugging this into $L[y]$; to do this, we calculuate $y_p' = -t\ln(t)e^{-t} + (\ln(t) + 1)e^{-t}$ and $y_p'' = -(y_p)' + \frac{1}{t}e^{-t} - (\ln(t) + 1)e^{-t}$. Then

$$\begin{aligned} L[y_p] &= y_p'' + 2y_p' + y_p \\ &= (\frac{1}{t}e^{-t} - (\ln(t)+1)e^{-t}) + (-t\ln(t)e^{-t} + (\ln(t)+1)e^{-t}) + t\ln(t)e^{-t} \\ &= t^{-1}e^{-t} = g(t). \end{aligned}$$

Thus the equation and forcing are $y'' + 2y' + y = t^{-1}e^{-t}$.

c. [3 points] If you were finding, by hand, the general solution given in **(b)**, what method or methods could you use? In these methods, what form do you guess for the solution?

Solution: Because the problem is linear, we know that we will be finding the solution to the complementary homogeneous problem and then finding a particular solution. Because it is constant-coefficient, the former will always be done by finding the solution to the eigenvalue problem obtained by looking for solutions of the form $y = e^{\lambda t}$ (or, $\mathbf{x} = \mathbf{v}e^{\lambda t}$ for the equivalent system of two first-order equations). The forcing term (and particular solution) do not admit use of the method of undertermined coefficients, so we would use variation of parameters to guess $y_p = u_1(t)e^{-t} + u_2(t)te^{-t}$.