7. [15 points] In our lab on lasers, we considered a linearization of the nonlinear model for the population inversion N and light intensity P. A critical point of the nonlinear system is $(N, P) = (1, A - 1)$, and linearizing the system near this gives the linear system

$$u' = -\gamma(Au + v), \qquad v' = (A - 1)u,$$

where γ and A are constants.

a. [5 points] Rewrite this as a single, second-order equation in v.

b. [5 points] Suppose that for some α and β your equation from (a) is $v'' + \alpha v' + \beta v = 0$. Under what conditions on α and β will the solution for v be underdamped? Write down two real-valued linearly independent solutions to the equation in this case.

c. [5 points] Now suppose that we force the underdamped equation given in (b) with the periodic forcing term $f(t) = \cos(\omega t)$. Sketch a graph of the steady state solution of the problem. Explain why your graph has the form it does. If ω changes from very small to very large values, how would you expect your sketch to change? Explain.