2. [14 points] Fill in the missing portions of each of the following transforms. Briefly explain how you obtain your work.

a. [7 points] $\mathcal{L}\{\begin{cases} 0, & 0 \le t < 1 \\ 1, & 1 \le t < 5 \\ e^{-(t-5)}, & t \ge 5 \end{cases}\} = \frac{1}{s}(e^{-s} - e^{-5s}) +$ $\underline{\quad e^{-5s}\frac{1}{s+1} \quad}$

Solution: Letting f be the indicated funcion and breaking the integral on the piecewise definitions, we have

$$\mathcal{L}\{f\} = \int_1^5 e^{-st}\,dt + \int_5^\infty e^{-(t-5)}e^{-st}\,dt.$$

The first of these integrals gives the terms provided. The second is

$$\int_5^\infty e^5 e^{-(s+1)t}\,dt = \lim_{b\to\infty}(-\frac{e^5}{s+1})e^{-(s+1)t}\Big|_{t=5}^{t=b} = \frac{e^5}{s+1}e^{-5s-5} = \frac{e^{-5s}}{s+1}.$$

b. [7 points] $\mathcal{L}^{-1}\{\frac{1}{s(s+1)(s^2+1)}\} = 1 - \frac{1}{2}\cos(t) +$ $\underline{\quad -\frac{1}{2}\sin(t) - \frac{1}{2}e^{-t} \quad}$

Solution: We can rewrite this with partial fractions:

$$\frac{1}{s(s+1)(s^2+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}.$$

The inverse transforms of these will be, in order, A, Be^{-t}, $C\cos(t)$, and $D\sin(t)$. Thus from the provided partial answer we know that $A = 1$ and $C = -\frac{1}{2}$. Then, clearing the denominators and using these values, we have

$$1 = (s+1)(s^2+1) + Bs(s^2+1) + (-\frac{1}{2}s + D)s(s+1).$$

If $s = -1$, we have $1 = -2B$, so that $B = -\frac{1}{2}$. If $s = 1$, we have $1 = 4-1+(-\frac{1}{2}+D)(2) = 2 + 2D$. Thus $D = -\frac{1}{2}$, and we have the indicated result.