2. [14 points] Fill in the missing portions of each of the following transforms. Briefly explain how you obtain your work.

a. [7 points] \( \mathcal{L}\left\{ \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t < 5 \\ e^{-(t-5)}, & t \geq 5 \end{cases} \right\} = \frac{1}{s} (e^{-s} - e^{-5s}) + \frac{e^{-5s}}{s+1} \)

**Solution:** Letting \( f \) be the indicated function and breaking the integral on the piecewise definitions, we have

\[ \mathcal{L}\{f\} = \int_1^5 e^{-st} \, dt + \int_5^\infty e^{-(t-5)} e^{-st} \, dt. \]

The first of these integrals gives the terms provided. The second is

\[ \int_5^\infty e^5 e^{-(s+1)t} \, dt = \lim_{b \to \infty} \left( \frac{e^5}{s+1} e^{-(s+1)t} \right) \bigg|_{t=5}^{t=b} = \frac{e^5}{s+1} e^{-5s-5} = \frac{e^{-5s}}{s+1}. \]

b. [7 points] \( \mathcal{L}^{-1}\left\{ \frac{1}{s(s+1)(s^2+1)} \right\} = 1 - \frac{1}{2} \cos(t) + \frac{1}{2} \sin(t) - \frac{1}{2} e^{-t} \)

**Solution:** We can rewrite this with partial fractions:

\[ \frac{1}{s(s+1)(s^2+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}. \]

The inverse transforms of these will be, in order, \( A, Be^{-t}, C \cos(t), \) and \( D \sin(t) \). Thus from the provided partial answer we know that \( A = 1 \) and \( C = -\frac{1}{2} \). Then, clearing the denominators and using these values, we have

\[ 1 = (s+1)(s^2+1) + Bs(s^2+1) + (-\frac{1}{2} s + D)s(s+1). \]

If \( s = -1 \), we have \( 1 = -2B \), so that \( B = -\frac{1}{2} \). If \( s = 1 \), we have \( 1 = 4 - 1 + (-\frac{1}{2} + D)(2) = 2 + 2D \). Thus \( D = -\frac{1}{2} \), and we have the indicated result.