**3**. [14 points] Find each of the following, as indicated.

**a**. [7 points] If a function $f(t)$ has the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$, use the integral definition of the Laplace transform to find the transform $\mathcal{L}\{\int_0^t f(x)\,dx\}$ in terms of $F(s)$. *(You may assume that $\int_0^\infty f(x)\,dx = L$, a finite value.)*

*Solution:* We have, integrating by parts with $u = \int_0^t f(x)\,dx$ (so that $u' = f(t)$) and $v' = e^{-st}$ (so that $v = -\frac{1}{s}e^{-st}$),

$$\begin{aligned}\mathcal{L}\{\int_0^T f(t)\,dt\} &= \int_0^\infty (\int_0^t f(x)\,dx)\,e^{-st}\,dt \\ &= -\frac{1}{s}\lim_{b\to\infty}(\int_0^t f(x)\,dx)\,e^{-st}\Big|_{t=0}^{t=b} + \frac{1}{s}\int_0^\infty f(t)e^{-st}\,dt \\ &= -\frac{1}{s}\int_0^0 f(x)\,dx + \lim_{b\to\infty}\frac{1}{s}L\,e^{-sb} + \frac{1}{s}F(s).\end{aligned}$$

The first term on the right-hand side is an integral over an interval of zero length, and so is zero, and in the limit as $b \to \infty$, the second vanishes because of the negative exponential $e^{-sb}$. Thus we have $\mathcal{L}\{\int_0^t f(x)\,dx\} = \frac{1}{s}\,F(s)$.

**b**. [7 points] Find an explicit expression for $Y = \mathcal{L}\{y\}$ if $y''' + 3y = t^2e^{-4t} - e^{-2t}\cos(5t)$. *(Note that you are not asked to solve the differential equation.)*

*Solution:* Because of the linearity of the transform, we can calculate the transform of each term separately. We have $\mathcal{L}\{y'''\} = s^3Y - s^2y(0) - sy'(0) - y''(0)$ and $\mathcal{L}\{3y\} = 3Y$. To find $\mathcal{L}\{t^2e^{-4t}\}$, we note that $\mathcal{L}\{e^{-4t}\} = \frac{1}{s+4}$, so that, by rule (B), $\mathcal{L}\{t^2e^{-4t}\} = \frac{d^2}{ds^2}\frac{1}{s+4} = \frac{2}{(s+4)^3}$. To find $\mathcal{L}\{e^{-2t}\cos(5t)\}$, we use rule (C) and the transform $\mathcal{L}\{\cos(5t)\} = \frac{s}{s^2+25}$ to get $\mathcal{L}\{e^{-2t}\cos(5t)\} = \frac{s+2}{(s+2)^2+25}$. Putting these all together and solving for $Y$, we have

$$Y = \frac{s^2y(0) + sy'(0) + y''(0)}{s^3+3} + \frac{2}{(s+4)^3(s^3+3)} - \frac{s+2}{((s+2)^2+25)(s^3+3)}.$$