4. [15 points] Find explicit, real-valued solutions for each of the following, as indicated. Do **not** use Laplace transform techniques on this problem.

a. [8 points] Find the general solution to $y'' + 2y' + 4y = e^{-t} + t^2$.

Solution: We first look for the general solution in the form $y = y_c + y_p$. For y_c, the guess $y = e^{rt}$ gives $r^2 + 2r + 4 = (r + 1)^2 + 3 = 0$, so that $r = -1 \pm \sqrt{3}\,i$, and $y_c = c_1 e^{-t}\cos(\sqrt{3}t) + c_2 e^{-t}\sin(\sqrt{3}t)$.

To find y_p, we use the method of undetermined coefficients, looking for each of the forms on the right-hand side separately. For the e^{-t} term, we guess $y = ae^{-t}$, so that $(1 - 2 + 4)a = 1$, and $a = \frac{1}{3}$. For the t^2 term, we guess $y = a_0 + a_1 t + a_2 t^2$, so that $2a_2 + 2(2a_2 t + a_1) + 4(a_2 t^2 + a_1 t + a_0) = t^2$. Collecting powers of t, we have $a_2 = \frac{1}{4}$, $a_1 = -\frac{1}{4}$, and $a_0 = 0$.

The general solution to the problem is therefore

$$y = c_1 e^{-t}\cos(\sqrt{3}t) + c_2 e^{-t}\sin(\sqrt{3}t) + \frac{1}{3}e^{-t} + \frac{1}{4}t^2 - \frac{1}{4}t.$$

b. [7 points] Solve $y'' + 5y' + 4y = 3\cos(2t)$, $y(0) = 0$, $y'(0) = 1$

Solution: We look for the general solution as $y = y_c + y_p$. For y_c we have $y = e^{rt}$, and $r^2 + 5r + 4 = (r + 4)(r + 1) = 0$, so that $y_c = c_1 e^{-4t} + c_2 e^{-t}$.

For y_p we use the method of undetermined coefficients guess $y_p = a\cos(2t) + b\sin(2t)$. Plugging in, we have

$$-4a\cos(2t) - 4b\sin(2t) - 10a\sin(2t) + 10b\cos(2t) + 4a\cos(2t) + 4b\sin(2t) = 3\cos(2t).$$

Collecting terms in $\cos(2t)$ and $\sin(2t)$, we have $10b = 3$, and $-10a = 0$; thus $a = 0$ and $b = \frac{3}{10}$. Our general solution is therefore

$$y = c_1 e^{-4t} + c_2 e^{-t} + \frac{3}{10}\sin(2t).$$

Applying the initial conditions, we have $y(0) = c_1 + c_2 = 0$, and $y'(0) = -4c_1 - c_2 + \frac{3}{5} = 1$. Substituting the first into the second, we have $3c_2 = \frac{2}{5}$, so that $c_2 = \frac{2}{15}$ and $c_1 = -\frac{2}{15}$. Thus

$$y = -\frac{2}{15}\,e^{-4t} + \frac{2}{15}\,e^{-t} + \frac{3}{10}\sin(2t).$$