5. [14 points] Consider the operators $T[y] = yy'' + 2y^2y'$ and $U[y] = t^2y'' - ty' - 3y$.

a. [9 points] Show that T is nonlinear while U is linear.

Solution: We note that

$$T[cy] = cy(cy'') - 2c^2y^2(cy') = c^2(yy'' - 2cy^2y') \neq cT[y]$$

(because $cT[y] = c(yy'' + 2y^2y')$). Therefore T is not linear. (Other possible arguments include the calculations $T[y_1 + y_2] \neq T[y_1 + y_2]$, and $T[c_1y_1 + c_2y_2] \neq c_1T[y_1] + c_2T[y_2]$.) However,

$$\begin{aligned} U[c_1y_1 + c_2y_2] &= t^2(c_1y_1'' + c_2y_2'') - t(c_1y_1' + c_2y_2') - 3(c_1y_1 + c_2y_2) \\ &= c_1(t^2y_1'' - ty_1' - 3y_1) + c_2(t^2y_2'' - ty_s' - 3y_2) \\ &= c_1U[y_1] + c_2U[y_2]. \end{aligned}$$

Thus U is linear.
Alternately, we said that a linear operator may also be written in the form $L = D^2 + p(t)D + q(t)$, so that $L[y] = D^2[y] + p(t)D[y] + q(t)y$. For T, we have

$$T[y] = yy'' + 2y^2y' = yD^2[y] + 2y^2D[y] = y(D^2[y] + 2yD[y]),$$

which is not in the form of a linear operator, while

$$U[y] = t^2(D^2[y] - t^{-1}D[y] - 3y) = t^2(D^2 - t^{-1}D - 3t^{-2})[y],$$

so that the operator $U = t^2D^2 - tD - 3$. Note that this isn't in the form of L; it is an easy generalization from that, but isn't quite consistent with our understanding. It's best to make the argument above, or to consider the operator in the context of a differential equation: the equation $U[y] = 0$ can be written as $L[y] = 0$, with $L = D^2 - t^{-1}D - 3t^{-2}$, a linear operator.

b. [5 points] Show that $y_1 = t^{-1}$ and $y_2 = t^3$ constitute a fundamental set of solutions to the equation $U[y] = 0$. What is the general solution to $U[y] = 0$?
(You may assume that $t > 0$.)

Solution: A fundamental solution set to a linear second-order homogeneous equation consists of two linearly independent solutions. Note that $U[y_1] = t^2(2t^{-3}) - t(-t^{-2}) - 3t^{-1} = 3t^{-1} - 3t^{-1} = 0$, and $U[y_2] = t^2(6t) - t(3t^2) - 3t^3 = 6t^3 - 6t^3 = 0$, so y_1 and y_2 are solutions to $U[y] = 0$. Then

$$W[y_1, y_2] = \begin{vmatrix} t^{-1} & t^3 \\ -t^{-2} & 3t^2 \end{vmatrix} = 3t + t = 4t \neq 0,$$

so the solutions are linearly independent. Thus the general solution is $y = c_1t^{-1} + c_2t^3$.