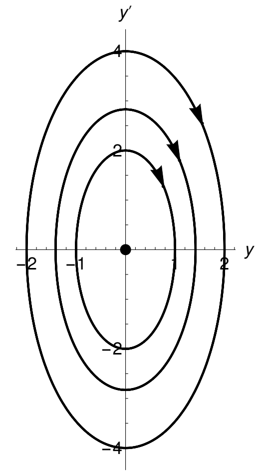
6. [13 points] Consider the phase portrait shown to the right, which shows the phase portrait for a linear, second-order, constant coefficient, homogeneous differential equation $L[y] = 0$.



a. [7 points] Write a differential equation that could give this phase portrait. Explain how you obtain your solution, and why is it correct.

Solution: We note that the phase portrait shows a center, that is, trajectories are simple closed loops. This suggests that the solutions are sines and cosines, so that we should have an equation $y'' + \omega_0^2 y = 0$. Solutions to this are $y_1 = \cos(\omega_0 t)$ and $y_2 = \sin(\omega_0 t)$, so that the phase plane trajectories are given by $\mathbf{x}_1 = c_1 \begin{pmatrix} \cos(\omega_0 t) \\ -\omega_0 \sin(\omega_0 t) \end{pmatrix}$ and $\mathbf{x}_2 = c_2 \begin{pmatrix} \sin(\omega_0 t) \\ \omega_0 \cos(\omega_0 t) \end{pmatrix}$. We note that the vertical stretch of the shown trajectories appears to be twice that of the horizontal, so guess that $\omega_0 = 2$. Our equation is therefore $y'' + 4y = 0$.

b. [6 points] Suppose that we add a forcing term $f(t) = \cos(15t/8)$ to the equation, so that we are solving $L[y] = f(t)$. Sketch an approximate solution curve with $y(0) = 0$, $y'(0) = 1$. Explain why your solution appears as it does.

Solution: Note that the forcing frequency $\omega = 15/8$ is close to the natural frequency of the system, $\omega_0 = 2$. So we expect to see at least a mild beats phenomenon. This is shown in the graph below.