7. [15 points] In our lab on lasers, we considered a linearization of the nonlinear model for the population inversion N and light intensity P. A critical point of the nonlinear system is $(N, P) = (1, A - 1)$, and linearizing the system near this gives the linear system

$$u' = -\gamma(Au + v), \qquad v' = (A - 1)u,$$

where γ and A are constants.

a. [5 points] Rewrite this as a single, second-order equation in v.

Solution: Note that, from the second equation, $u = \frac{1}{A-1}\, v'$. Plugging this into the first equation, we have

$$\frac{1}{A-1}\, v'' + \frac{\gamma A}{A-1}\, v' + \gamma v = 0,$$

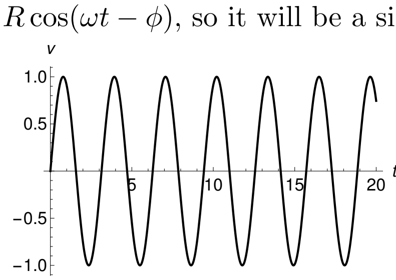
or $v'' + \gamma A\, v' + \gamma(A - 1)v = 0$.

b. [5 points] Suppose that for some α and β your equation from (a) is $v'' + \alpha v' + \beta v = 0$. Under what conditions on α and β will the solution for v be underdamped? Write down two real-valued linearly independent solutions to the equation in this case.

Solution: We note that solutions to this equation are $v = e^{rt}$ with $r = -\frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 4\beta}$. This will give underdamping if $\alpha^2 - 4\beta < 0$, that is, if $\alpha^2 < 4\beta$. In terms of the constants we obtained in (a), this is $\gamma^2 A^2 < 4\gamma(A - 1)$, or $\gamma < 4\frac{A-1}{A}^2$. The two solutions are $y_1 = e^{-\mu t}\cos(\nu t)$ and $y_2 = e^{-\mu t}\sin(\nu t)$, where $\mu = \frac{\alpha}{2} = \frac{\gamma A}{2}$ and $\nu = \frac{1}{2}\sqrt{4\beta - \alpha^2} = \frac{1}{2}\sqrt{4\gamma(A - 1) - \gamma^2 A^2}$.

c. [5 points] Now suppose that we force the underdamped equation given in (b) with the periodic forcing term $f(t) = \cos(\omega t)$. Sketch a graph of the steady state solution of the problem. Explain why your graph has the form it does. If ω changes from very small to very large values, how would you expect your sketch to change? Explain.

Solution: The steady state solution will be the response to the forcing, because (as we see in (b)) the non-forced response decays to zero. Because y_1 and y_2 do not have the same form as $f(t)$, we know that the steady state (particular) solution will have the form $v_p = a\cos(\omega t) + b\sin(\omega t) = R\cos(\omega t - \phi)$, so it will be a simple sinusoid:



If we vary ω, we expect that the frequency of the solution will change, and that its amplitude will also change. A reasonable guess is that the amplitude will initially increase, obtain a local maximum at an ω near $\nu = \frac{1}{2}\sqrt{4\beta - \alpha^2} = \frac{1}{2}\sqrt{4\gamma(A - 1) - \gamma^2 A^2}$, and then decay to zero as ω becomes very large.