1. [15 points] Find explicit, real-valued solutions to each of the following, as indicated. For this problem, **DO NOT use Laplace transforms**.

a. [7 points] Find the general solution to $2U''(t) + 12U'(t) + 16U(t) = 12e^{4t}$.

Solution: This problem is nonhomogeneous, linear, and constant-coefficient. The general solution will be $U = U_c + U_p$, where U_c is the general solution to the complementary homogeneous problem. For this we look for a solution $U = e^{\lambda t}$. Plugging in to the homogeneous equation, we have $2\lambda^2 + 12\lambda + 16 = 2(\lambda + 4)(\lambda + 2) = 0$. Thus $\lambda = -4$ or $\lambda = -2$, so that U_c is given by $U_c = c_1 e^{-4t} + c_2 e^{-2t}$.
Then, to find Up, we use undertermined coefficients and look for a solution of the form $U_p = Ae^{4t}$. Plugging into the differential equation, we have $A(2(16) + 12(4) + 16)e^{4t} = A(96)e^{4t} = 12e^{4t}$, so that $A = \frac{12}{96} = \frac{1}{8}$, and

$$U = c_1 e^{-4t} + c_2 e^{-2t} + \frac{1}{8} e^{4t}.$$

b. [8 points] Find the solution to the initial value problem $y''(t) + 6y'(t) + 9y(t) = 3e^{-3t}$, $y(0) = 0$, $y'(0) = 1$.

Solution: Again, the general solution will be $y = y_c + y_p$. For y_c, the characteristic equation is $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0$, so $\lambda = -3$, twice, and $y_c = c_1 e^{-3t} + c_2 t e^{-3t}$.
For y_p, we would use the method of undetermined coefficients and guess $y_p = ae^{-3t}$; however, this is part of y_c, so we must multiply by t^2 in order for that not to be the case. Thus we guess $y_p = at^2 e^{-3t}$. Then $y_p' = (-3at^2 + 2at)e^{-3t}$, and $y_p'' = (9at^2 - 6at + 2a)e^{-3t}$, so that, plugging in, we have

$$\begin{aligned}(9at^2 - 6at + 2a)e^{-3t} + 6(-3at^2 + 2at)e^{-3t}) + 9at^2)e^{-3t} &= 3e^{-3t}\\ 2ae^{-3t} &= 3e^{-3t},\end{aligned}$$

and $y_p = \frac{3}{2} t^2 e^{-3t}$. The general solution is therefore $y = c_1 e^{-3t} + c_2 t e^{-3t} + \frac{3}{2} t^2 e^{-3t}$. Applying the initial conditions, we have $y(0) = c_1 = 0$ and $y'(0) = c_2 = 1$. Thus $y = te^{-3t} + \frac{3}{2} t^2 e^{-3t}$.