2. [15 points] Find explicit, real-valued solutions to each of the following, as indicated. For this problem, **DO use Laplace transforms**.

a. [7 points] Find the solution to the initial value problem $y'' + 4y' + 20y = 0$, $y(0) = 1$, $y'(0) = 5$.

Solution: Taking the Laplace transform of both sides of the equation, we have $\mathcal{L}\{y'' + 4y' + 20y\} = 0$, so that, with $Y = \mathcal{L}\{y\}$,

$$s^2Y - s - 5 + 4(sY - 1) + 20Y = 0,$$

so that

$$Y = \frac{s+9}{s^2+4s+20} = \frac{(s+2)+7}{(s+2)^2+16}.$$

Taking the inverse transform, we have

$$y = \mathcal{L}^{-1}\{\frac{s+2}{(s+2)^2+16}\} + \mathcal{L}^{-1}\{\frac{7}{(s+2)^2+16}\} = e^{-2t}\cos(4t) + \frac{7}{4}e^{-2t}\sin(4t).$$

b. [8 points] Find the solution to the initial value problem $y'' + 3y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 0$.

Solution: Proceding as above, the forward transform gives

$$s^2Y + 3sY + 2Y = \frac{1}{s+1},$$

so that

$$Y = \frac{1}{(s+1)^2(s+2)}.$$

To find the inverse transform, we use partial fractions, letting

$$\frac{1}{(s+1)^2(s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}.$$

Clearing the denominator,

$$1 = A(s+1)(s+2) + B(s+2) + C(s+1)^2.$$

Taking $s = -1$ and $s = -2$, we find $B = 1$ and $C = 1$. Then, if $s = 0$, $1 = 2A + 3$, and $A = -1$.

$$y = \mathcal{L}^{-1}\{-\frac{1}{(s+1)}\} + \mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\} + \mathcal{L}^{-1}\{\frac{1}{(s+2)}\} = -e^{-t} + te^{-t} + e^{-2t}.$$