3. [14 points] Suppose that $L[y] = y'' + p(t)y' + q(t)y$. *(Note that $L[y]$ here is a differential operator, not the Laplace transform $\mathcal{L}\{y\}$.)*

a. [7 points] If $L[t^2] = 2 + 2tp(t) + t^2q(t) = 0$ and $L[t^2\ln(t)] = (2\ln(t) + 3) + (2t\ln(t) + t)p(t) + t^2\ln(t)q(t) = 0$, which, if any, of the following functions y are solutions to $L[y] = 0$ on the domain $t > 0$? Which, if any, give a general solution on this domain? Why? (In these expressions, c_1 and c_2 are real constants.)

$y_1 = 5t^2$	$y_2 = 5t^2(1 + 2\ln(t))$	$y_3 = c_1t^2 + c_2t^2\ln(t)$
$y_4 = -t^2\ln(t)$	$y_5 = t^4\ln(t)$	$y_6 = c_1t^2(1 + \ln(t))$
$y_7 = t^{-2}\ln(t)$	$y_8 = W[t^2, t^2\ln(t)] = t^3$	$y_9 = c_1(5t^2 - 2c_2\ln(t))$

Solution: Note that the Wronskian $W[t^2, t^2\ln(t)]$ is correctly given in y_8: $W[t^2, t^2\ln(t)] = t^2(2t\ln(t) + t) - t^3\ln(t) = t^3$. Thus these two functions are linearly independent, and a general linear combination of the two will give the general solution. Further, because L is linear, any linear combination of the two will be a solution. Thus all of y_1, y_2, y_3, y_4, and y_6 are solutions. Those solutions with an arbitrary constant multiplying each of linearly independent solutions are general solutions; this is only y_3.

b. [7 points] Now suppose that $p(t) = 2$ and $q(t) = 10$, and let $L[y] = y'' + 2y' + 10y = g(t)$. For what $g(t)$ will the steady state solution to this problem be constant? Solve your equation with this $g(t)$ and explain how your solution confirms that your $g(t)$ is correct.

Solution: We will get a constant steady state whenever $g(t) = k \in \mathbb{R}$. Note that in this case the characteristic equation is $\lambda^2 + 2\lambda + 10 = (\lambda + 1)^2 + 9 = 0$, so $\lambda = -1 \pm 3i$, and $y_c = c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t)$. The particular solution is $y_p = k/10$. Thus $y = c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t) + \frac{k}{10}$, and as $t \to \infty$ we see that $y_c \to 0$ and $y \to \frac{k}{10}$, a constant.