4. [15 points] In lab 2 we considered the van der Pol oscillator, modeled by the equation, $x'' + \mu(x^2 - 1)x' + x = 0$. Recall that there is a single critical point for this system, $x = 0$, near which we may model the behavior of the oscillator with the linear equation $x'' - \mu x' + x = 0$.

a. [5 points] Suppose that $\mu = -1$. Find the amplitude of the solution to the linear problem with initial condition $x(0) = 2$, $x'(0) = 3$. What is the time after which this amplitude never exceeds some value a_0?

Solution: If $\mu = -1$, the characteristic equation for the problem is $\lambda^2 + \lambda + 1 = 0$, so $\lambda = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The general solution to the problem is therefore $x = c_1 e^{-t/2}\cos(\frac{\sqrt{3}}{2}\,t) + c_2 e^{-t/2}\sin(\frac{\sqrt{3}}{2}\,t)$. Applying the initial condition, $x(0) = c_1 = 2$, and $x'(0) = -1 + \frac{\sqrt{3}}{2}\,c_2 = 3$, so $c_2 = 8/\sqrt{3}$. The solution to the initial value problem is $x = 2e^{-t/2}\cos(\sqrt{3}\,t) + (8/\sqrt{3})e^{-t/2}\sin(\sqrt{3}\,t)$.

The amplitude of this solution is $R = \sqrt{4 + 64/3}\,e^{-t/2} = \sqrt{76/3}\,e^{-t/2}$, so this is less than a_0 when $e^{-t/2} < a_0/\sqrt{76/3}$, or, when $t > 2\ln(\sqrt{76/3}/a_0)$.

b. [5 points] Suppose we force the linear system with an oscillatory input, so that we are considering $x'' - \mu x' + x = \cos(\omega t)$ (and $\omega \neq 0$). For what values of μ will the system have an oscillatory steady-state solution with frequency ω?

Solution: Note that the characteristic equation for this problem is $\lambda^2 - \mu\lambda + 1 = 0$, so that $\lambda = \frac{1}{2}\mu \pm \frac{1}{2}\sqrt{\mu^2 - 4}$. Thus if $\mu > 0$ we are guaranteed at least one positive root, and the homogeneous solution will not decay. If $\mu = 0$, we have no damping and the solution to the problem will either be $y = c_1\cos(2t) + c_2\sin(2t) + R\cos(\omega t - \delta)$ or, if $\omega = 2$, a growing solution. In either case we do not have an oscillatory steady-state solution with frequency ω.

If $-2 < \mu < 0$, the homogeneous solution will be a decaying oscillatory solution, and if $\mu \leq -2$ it will be a decaying exponential (and te^{-2t} if $\mu = -2$). In either case the steady-state solution will be $y_p = a\cos(\omega t) + b\sin(\omega t)$, and hence be oscillatory with frequency ω.

Thus, we need $\mu < 0$.

c. [5 points] Suppose that, for some choice of μ, the system $x'' - \mu x' + x = \cos(\omega t)$ has an oscillatory steady-state solution, and that the gain function $G(\omega)$ for this solution is shown to the right, below. If the steady-state solution to the problem is $y_{ss} = R\cos(t - \pi/2)$, what are the R in the solution, and ω in the forcing term? Why?

Solution: The steady state solution to the problem has frequency ω, so $\omega = 1$. Then, if $\omega = 1$, we see from the gain function that $R = 1$.