4. [15 points] In lab 2 we considered the van der Pol oscillator, modeled by the equation, \( x'' + \mu(x^2 - 1)x' + x = 0 \). Recall that there is a single critical point for this system, \( x = 0 \), near which we may model the behavior of the oscillator with the linear equation \( x'' - \mu x' + x = 0 \).

a. [5 points] Suppose that \( \mu = -1 \). Find the amplitude of the solution to the linear problem with initial condition \( x(0) = 2, x'(0) = 3 \). What is the time after which this amplitude never exceeds some value \( a_0 \)?

**Solution:** If \( \mu = -1 \), the characteristic equation for the problem is \( \lambda^2 + \lambda + 1 = 0 \), so \( \lambda = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3} \). The general solution to the problem is therefore \( x = c_1 e^{-t/2} \cos(\frac{\sqrt{3}}{2} t) + c_2 e^{-t/2} \sin(\frac{\sqrt{3}}{2} t) \). Applying the initial condition, \( x(0) = c_1 = 2 \), and \( x'(0) = -1 + \frac{\sqrt{3}}{2} c_2 = 3 \), so \( c_2 = 8/\sqrt{3} \). The solution to the initial value problem is \( x = 2e^{-t/2} \cos(\sqrt{3}t) + (8/\sqrt{3})e^{-t/2} \sin(\sqrt{3}t) \).

The amplitude of this solution is \( R = \sqrt{4 + 64/3} e^{-t/2} = \sqrt{76/3} e^{-t/2} \), so this is less than \( a_0 \) when \( e^{-t/2} < a_0/\sqrt{76/3} \), or, when \( t > 2 \ln(\sqrt{76/3}/a_0) \).

b. [5 points] Suppose we force the linear system with an oscillatory input, so that we are considering \( x'' - \mu x' + x = \cos(\omega t) \) (and \( \omega \neq 0 \)). For what values of \( \mu \) will the system have an oscillatory steady-state solution with frequency \( \omega \)?

**Solution:** Note that the characteristic equation for this problem is \( \lambda^2 - \mu \lambda + 1 = 0 \), so that \( \lambda = \frac{1}{2} \mu \pm \frac{1}{2} \sqrt{\mu^2 - 4} \). Thus if \( \mu > 0 \) we are guaranteed at least one positive root, and the homogeneous solution will not decay. If \( \mu = 0 \), we have no damping and the solution to the problem will either be \( y = c_1 \cos(2t) + c_2 \sin(2t) + R \cos(\omega t - \delta) \) or, if \( \omega = 2 \), a growing solution. In either case we do not have an oscillatory steady-state solution with frequency \( \omega \).

If \(-2 < \mu < 0 \), the homogeneous solution will be a decaying oscillatory solution, and if \( \mu \leq -2 \) it will be a decaying exponential (and \( te^{-2t} \) if \( \mu = -2 \)). In either case the steady-state solution will be \( y_p = a \cos(\omega t) + b \sin(\omega t) \), and hence be oscillatory with frequency \( \omega \).

Thus, we need \( \mu < 0 \).

c. [5 points] Suppose that, for some choice of \( \mu \), the system \( x'' - \mu x' + x = \cos(\omega t) \) has an oscillatory steady-state solution, and that the gain function \( G(\omega) \) for this solution is shown to the right, below. If the steady-state solution to the problem is \( y_{ss} = R \cos(t - \pi/2) \), what are the \( R \) in the solution, and \( \omega \) in the forcing term? Why?

**Solution:** The steady state solution to the problem has frequency \( \omega \), so \( \omega = 1 \). Then, if \( \omega = 1 \), we see from the gain function that \( R = 1 \).