5. [14 points] In each of the following we consider a linear, second order, constant coefficient operator L, so that $L[y] = 0$ is a homogenous differential equation. (Note, however, that the operator L may be different in each of the parts below.) Let $y(0) = y_0$ and $y'(0) = v_0$, where y_0 and v_0 are real numbers.

a. [7 points] If the general solution to the equation $L[y] = 0$ is $y = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t)$, what is the Laplace transform $Y(s) = \mathcal{L}\{y(t)\}$?

Solution: The general solution tells us that the two roots of the characteristic polynomial are $\lambda = -3 \pm 2i$. Thus the characteristic polynomial is $(\lambda+3)^2 + 4 = \lambda^2 + 6\lambda + 13$, and so $L[y] = y'' + 6y' + 13y$. The forward transform of this will be $\mathcal{L}\{y'' + 6y' + 13y\} = -sy_0 - v_0 + s^2Y - 6y_0 + 6sY + 13Y = 0$, and

$$Y = \frac{sy_0 + v_0 + 6y_0}{s^2 + 6s + 13}.$$

b. [7 points] Now suppose that we are solving $L[y] = k$, for some constant k, and that y_0 and v_0 are both zero (so that $y(0) = y'(0) = 0$). If $Y(s) = \mathcal{L}\{y(t)\}$ is $Y = \frac{k}{s(s+3)(s+4)}$, what is the differential equation we are solving, and the general solution to the complementary homogeneous problem? Explain how you know your answer is correct.

Solution: Note that $\mathcal{L}\{k\} = \frac{k}{s}$. Therefore the operator L is $L = (D+3)(D+4)$, to get the quadratic term multiplying s in the denominator of Y. That is, $L = D^2 + 7D + 12$, so that $L[y] = y'' + 7y' + 12y$, and the general solution to the problem is $y = c_1e^{-3t} + c_2e^{-4t}$.