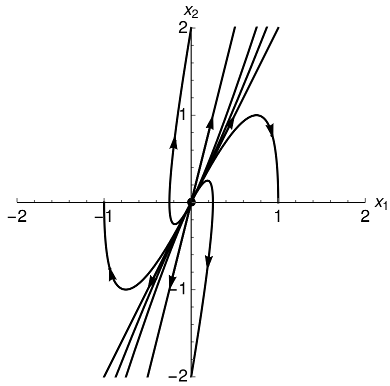
6. [15 points] Each of the following concerns a linear, second order, constant coefficient differential equation $y'' + py' + qy = 0$.

a. [7 points] If the general solution to the problem is $y = c_1e^{2t} + c_2e^{4t}$, sketch a phase portrait for the system.

Solution: In matrix form, the general solution to the system will be $\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix}$. Thus eigenvectors for the system will be $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, with $\lambda = 2$ and $\lambda = 4$, respectively. We therefore have the phase portrait shown.



b. [8 points] Now suppose that for some real-valued α, we have $p = 2\alpha$ and $q = 1$, so that we are considering $y'' + 2\alpha y' + y = 0$. For what values of α, if any

(i) do all solutions to the differential equation decay to zero?

(ii) are there solutions that do not decay to zero?

(iii) will the general solution be a decaying sinusoidal function?

Solution: We see that the characteristic polynomial is in this case $\lambda^2 + 2\alpha\lambda + 1 = 0$, so that $\lambda = -\alpha \pm \sqrt{\alpha^2 - 1}$. Thus: **(i)** The square root can never be larger in magnitude than α, so for all $\alpha > 0$ we will have solutions that decay to zero. **(ii)** If $\alpha \leq 0$, there will be solutions that do not decay to zero, and, in fact, if $\alpha < 0$ all solutions will diverge to infinity. **(iii)** If $|\alpha| < 1$, the square root will be imaginary, so that $\lambda = -\alpha \pm i\sqrt{1 - \alpha^2}$. In this case solutions if $0 < \alpha < 1$ we will have decaying sinusoidal solutions.